Chaos and fluctuations in quantum dots

A. Douglas Stone and Henrik Bruus

Applied Physics, Yale University, New Haven, CT 06520, USA

We consider the influence of chaotic classical dynamics on the spectrum and conducting properties of semiconductor quantum dots in the nearly-isolated (Coulomb blockade) regime. In this regime the amplitude of the Coulomb blockade oscillations are found to fluctuate strongly from peak to peak and as a function of magnetic field. Microscopic calculations are performed on Robnik's model, a class of billiards obtained by quadratic conformal transformations of the unit disk. It is known that small distortions of this system from cylindrical symmetry lead to fully chaotic classical dynamics and a quantum spectrum well-described by the Wigner-Dyson random matrix ensembles. We show that in the chaotic regime the levels are rapidly fluctuating as a function of shape and comment on the implications of these results for quantum dot superlattices. We derive a universal one-parameter distribution describing the fluctuations of the resonance amplitudes in the regime where a single thermally-broadened level dominates. Breaking time-reversal symmetry with a weak magnetic field is shown to change this distribution substantially, an effect which should be experimentally observable. We find excellent agreement between numerical results from Robnik's model and the random-matrix model in the chaotic limit, and the expected disagreement in the regular limit. This indicates that this fluctuation phenomenon can distinguish between regular and chaotic dot potentials.

1. Introduction

Beginning with the work of Reed et al. [1] and Meirav et al. [2] it has become fairly routine to fabricate semiconductor heterostructures with islands of trapped electrons ("quantum dots") and study the transport properties of individual islands. The similarity of these systems to artificial atoms has been widely noted and model calculations based on this analogy have been performed for electrons trapped within a two-dimensional confining potential both in the non-interacting and the interacting case (and often in the presence of a magnetic field). However, unlike natural atoms, the potential which confines each semiconductor quantum dot will vary from dot to dot due both to variations in the confinement mechanism (e.g. electrostatic gates, anisotropy etching, ion-damage) and to weak potential fluctuations arising from the randomly located donors. In fact there are several pieces of experimental evidence that potential fluctuations are playing an important role in the transport properties of such nanostructures at low temperatures once $kT < \Delta \epsilon$, where $\Delta \epsilon$ is the mean single-particle level-spacing of the dot. First and most dramatically the amplitude of the Coulomb-blockade oscillations fluctuates non-monotonically by as much as an order of magnitude between adjacent peaks [2-4] (see fig. 1). Second, the oscillation amplitude pattern is not well-reproduced upon thermal cycling [2], suggesting that small redistribution of charge can have a significant effect on this pattern. Third, even at low temperature this pattern is completely rearranged in an apparently random fashion by magnetic fields of order 20 mT [4] (see fig. 2). Finally, one-dimensional arrays of nominally identical quantum dots fail to show the miniband structure expected of 1d superlattice (expect in the high-magnetic field regime above 2T where Landau quantization suppresses the sensitivity to the confining potential) [5].

The goal of the present work is to develop a theory of the spectrum and eigenstates of quan-
conserve any constants of motion. Thus, we will consider models which generically have no constants of motion (other than energy) and in which spatial symmetries are partially or completely broken. The removal of conserved quantities in the classical motion will lead to a transition between regular and chaotic classical dynamics and it is now well-known that the quantum signature of this transition is a qualitative change in the statistical properties of the spectrum \([6,7]\) and eigenstates \([8]\). These statistical properties cannot be obtained by perturbation theory around the regular Hamiltonian, and it will be essential to consider directly an ensemble of irregular quantum dots with chaotic classical dynamics.

In previous work \([9]\) we considered the distribution of resonance widths of irregular quantum dots which we argued should determine the distribution of amplitude fluctuations of the Coulomb blockade oscillations in the regime where a single thermally broadened level dominates \((kT < \Delta E)\). We pointed out that these amplitude fluctuations were similar in origin to the well-studied Porter–Thomas fluctuations in the resonance widths of elastic compound nuclear scattering \([10]\), except that the measured statistic was the area under the resonance instead of simply its width. In analogy to the treatment of Porter–Thomas fluctuations we used a random-matrix theory ansatz to derive the form of the area distribution analytically. We then tested the ansatz by direct fitting of resonance data generated by numerical evaluation of the transmission coefficient through an ensemble of weakly-disordered quantum dots. We shall review these random-matrix arguments below. In this work we will consider both the spectrum and the resonance properties of irregular quantum dots within a new model based on a class of “quantum billiards” first investigated by Robnik \([11,12]\).

2. Robnik dots

Robnik introduced and studied the classical and quantum mechanics of the class of billiards
generated by area-preserving quadratic conformal transformations of the unit disk [11–13]. Employing this model to study irregular quantum dots has several advantages with respect to our earlier work. First, as Robnik showed [12], the spectrum and eigenstates for such billiards may be calculated by a very efficient numerical procedure summarized in section 3. This allows us to evaluate very simply the sensitivity of the spectrum to shape distortion (see fig. 3, below). Second, it is possible to obtain enough numerical data from the different energy levels to compare to our statistical theory without introducing any randomness at all into the model. Third, the distribution of reduced resonance widths (to be defined below) may be generated directly from knowledge of the eigenstates (with Neumann boundary conditions) using $R$-matrix theory and the “single-level” approximation [14], removing the necessity of the rather complicated fitting procedure employed in ref. [9]. Finally, within this model it is straightforward to study the change of this distribution under a transition from chaotic to regular dynamics. A minor disadvantage of all models based on billiards is that the infinitely sharp walls at the boundary are unrealistic as a description of semiconductor quantum dots and probably overestimates the sensitivity of the system to shape distortions. However, it should be remembered that we are primarily interested in describing experiments on lateral quantum dots which typically have of the order of 100 to 500 one-electron levels occupied in equilibrium and hence the confining potential

![Diagram](image)

Fig. 3. The positive parity energy levels 92–103 (in units of $\hbar^2/(2mR^2)$, $R$ being the radius) of the Robnik dot vs. the shape parameter $p$. The shape of the billiard is shown for the regular case ($p = 0$), at the point of transition to hard chaos ($p = p_c$), and well into the chaotic regime ($p = 0.6$).
at the Fermi-energy is rather steep and far away from e.g. the harmonic well approximation.

The Robnik model consists of all two-dimensional infinite potential wells with boundaries given by the conformal mapping \( w(z) = az + bz^2 \) of the unit circle. The area, \( A = \pi(a^2 + 2b^2) \), of the well is kept fixed at the value \( \pi \) to remove the trivial variation of the level density with area by choosing the parameterization \( a = \cos p \) and \( b = (\sin p)/\sqrt{2} \). The position of the boundary of the well for a given shape parameter \( p \) is then

\[
\begin{align*}
    x &= \cos p \cos \theta + \frac{1}{\sqrt{2}} \sin p \cos 2\theta, \\
    y &= \cos p \sin \theta + \frac{1}{\sqrt{2}} \sin p \sin 2\theta,
\end{align*}
\]

(1)

where \( \theta \) is the polar angle. Initially we shall require that the wave functions vanish on this boundary (Dirichlet boundary conditions); later in our formulation of R-matrix theory we will require a vanishing derivative at this boundary instead (Neumann boundary conditions). Choosing \( p = 0 \) describes the circular well of unit radius. In this case the (un-normalized) eigen-functions are just \( \phi_{q,n}(r, \theta) = J_q(k_{q,n}r) \cos n\theta \), \( J_q(k_{q,n}r) \sin n\theta \), where \( J_q \) is the Bessel function of order \( q \) and \( k_{q,n} \) is determined by the requirement that the wave function (or its derivative) vanish at \( r = 1 \), hence the label \( n \) refers to the \( n \)th zero of \( J_q \) (or of its derivative). At \( p = 0 \) the even and odd parity solutions are degenerate; as \( p \) increases the circular boundary gets progressively distorted and different \( \theta_{q,n} \) are mixed, but due to the conservation of mirror symmetry with respect to the \( x \) axis during the deformation, parity remains a good quantum number, and cosine and sine solutions never mix. For \( p = p_c = \arctan(1/\sqrt{8}) \approx 0.34 \) the curvature of the boundary at \( \theta = \pi \) vanishes and the billiard begins to develop a concave region and a transition to fully chaotic classical dynamics takes place [11]. At \( p = p_c = \arctan(1/\sqrt{2}) = 0.62 \) the boundary develops a cusp at \( \theta = \pi \) (i.e. \( dw/dz = 0 \)) and the mapping is no longer conformal; hence we consider the class of billiards generated by choosing \( 0 \leq p < p_c \). The shape of the billiard for \( p = 0 \), \( p_c \), and 0.6 is shown in fig. 3.

3. Spectra of irregular dots

The basic simplification of the Robnik model is that the Schrödinger equation of the irregular billiards may be solved by the change of variables defined by the inverse conformal transformation (back to the unit disk). In the new variables the Dirichlet (or Neumann) boundary conditions will again be imposed on the boundary of the unit disk and the only price paid is the appearance of an effective potential energy term related to \( f(r, \theta) = |dw/dz|^2 \), the Jacobian of the transformation. The presence of this potential energy term means that the functions \( \phi_{q,n} \) are no longer solutions but they provide a convenient complete basis for representing the true solutions since they already satisfy the boundary conditions. Robnik showed [12] that the form of \( f(r, \theta) \) yields an infinite-order secular equation for the eigenvalues at non-zero \( p \) which only requires evaluation of matrix elements of \( r^2 \) and \( r \cos \theta \) between the \( \phi_{q,n} \) which then can be reused for each \( p \). Truncation of the secular equation at finite order (in our calculations we typically include the lowest 500 \( \phi_{q,n} \)) allows straightforward calculation of the energy levels and wave functions of the lowest 150 levels as a function of shape parameter \( p \) with good accuracy.

In fig. 3 we plot the positive parity energy levels 92–103 as a function of \( p \). There are several points to note. The spectrum at \( p = p_c \) is clearly uncorrelated with that at \( p = 0 \), so even this minor distortion of the circle makes the eigenstates and energies of the perfect disk irrelevant for describing the levels near the hundredth (of course the very lowest levels do show much weaker and smoother shape-dependence). Second, there is a clear qualitative difference between the fully chaotic regime (the interval \( p_c < p < p_1 \)) and the regular/mixed regime (the interval \( 0 < p < p_c \)). In the chaotic regime we typically see several oscillations in the spectrum as a function of shape and large gaps at the level crossings. In the regular/mixed regime over a \( p \) interval of the same size we see essentially monotonic shape dependence except at level anti-crossings which have very small gaps due to
the weak inter-level repulsion (on the scale of fig. 3 these appear as level crossings until the crossing region is magnified). Since there are no remaining symmetries after a definite parity has been selected (in fig. 3, the positive), no true crossings may occur. Robnik [12] has discussed the interpretation of the $p$ dependence of the spectra near $p=0$ which may be understood perturbatively and in terms of compression or expansion of almost regular wave-functions. Such an analysis is not relevant to the chaotic regime and we propose instead to characterize the spectrum in this regime by the correlation function of the derivatives of the energy levels with respect to $p$. The half-width of this correlation function gives a measure of the sensitivity of the levels to shape distortion in the chaotic regime. We find that this half-width is $p_{ correlation} = 0.03$ for the mean energy of the interval shown in fig. 3. The size of $p_{ correlation}$ should clearly decrease with increasing energy as the shorter electron wavelength gives rise to stronger sensitivity to distortion; we find it decreases roughly linearly with energy varying from 0.045 to 0.020 between the 50th and 150th level. Details of this calculation will be given elsewhere [15].

4. Implications for quantum dot arrays

The results of fig. 3 have strong implications for transport experiments on arrays of coupled quantum dots. To understand such experiments qualitatively let us neglect fluctuations in the tunnel coupling, $I'$, between the dots and the coupling, $I_{ext}$, of the array to the leads. Initially we will consider the case where the array is well-isolated from the leads, so that $I_{ext}$ is the smallest energy in the problem, and we will neglect charging effects.

A very tempting model for these systems is to describe them as small quantum dot superlattices with allowed and forbidden energy bands created by resonant mixing of states between dots. For this model to be applicable the corresponding energy levels in each dot must differ by less than the inter-dot coupling $I$. Consider the simplest case of a 1d “mini-crystal” consisting of $N$ dots, as was studied in the experiment of ref. [5]. If we denote the energy-level spacing of a single dot by $\Delta \epsilon$, then, in the case of $I' \ll \Delta \epsilon$ and identical level sequences, each level of the sequence gives rise to a “band” of bandwidth $\sim I'$ containing $N$ discrete states of internal level spacing (mini-gaps) $\sim I'/N$. The gaps between the bands in this case will be $\sim \Delta \epsilon$, and if we now perform conductance measurements through this crystal at $kT < \Delta \epsilon$ we will observe thermally broadened tunneling resonances of width $kT$ spaced by $\Delta \epsilon$. If we could make $kT < I'/N$, we would see each resonance break up into $N$ closely space resonances separated by mini-gaps of this size. If, as in the recent experiment, we increase the coupling so that $I' \ll \Delta \epsilon$ then we will create overlapping hybridized bands which will still exhibit $N$ mini-gaps of order $I'/N$. These mini-gaps were expected to be the signature of a 1d mini-crystal in the experiment. In fact such mini-gaps were not observed in the experimental conductance measurements at zero magnetic field (although they were observed above 2 T) [5].

Consider now an alternative to the superlattice model, which is motivated by our calculations indicating that small shape distortions would cause each dot in isolation to have uncorrelated energy levels. An appropriate model for the array in this case would be a chain of sites with pseudo-random on-site energies (a many-orbital 1d Anderson model). If $I' \ll \Delta \epsilon$ the states of the coupled chain would be trivially localized on each dot, and the resonances of the total chain would likely be too narrow to observe. As the inter-dot couplings are increased so that $I' > \Delta \epsilon$ the localization length will grow as $(I' / \Delta \epsilon)^2$ and will rapidly exceed the length of the chain. In this case resonances should reappear but (neglecting charging effects) with typical spacing $\Delta \epsilon / N$ not $\Delta \epsilon$, and no mini-band structure should exist.

In the experiment of ref. [5] the device was fabricated such that $I_{ext} = I$ and the two parameters were not independently variable. Thus as $I$ was increased the chain of dots also became well-coupled to the leads, causing the charging energy to go to zero and making the above qualitative argument (which neglected charging
effects) applicable. Thus the absence of mini-gaps at low magnetic field may be explained by the hypothesis that shape fluctuations in each dot prevents resonances between dots in the array and hence band formation does not occur. In a sufficiently high magnetic field adiabatic edge-channels are formed and the shape-sensitivity of the levels is suppressed, leading to the band formation observed in the experiment above 2 T. This indirect argument suggests that quantum dots at this high occupancy are not well-described as identical “artificial atoms”. We now review our statistical theory for the distribution of the Coulomb blockade amplitude fluctuations and test it numerically in the context of the Robnik model.

5. Coulomb blockade resonance amplitudes and level widths

The large resonance amplitude fluctuations shown in fig. 1 are only observed at very low temperatures (typically below 500 mK in the experimental systems) when \( kT < \Delta \epsilon < e^2/C \), \( C \) being a characteristic capacitance. In the intermediate temperature regime, where \( \Delta \epsilon < kT < e^2/C \), experiments [2] show much weaker fluctuations, but an anomalous dependence of peak shape on temperature. All peaks narrow \( \sim kT \) as temperature is reduced, but while some peaks grow with decreasing temperature many others decrease in amplitude, contradicting the familiar behavior of thermally-broadened resonant tunneling peaks (for which the peak area would be preserved). Recently Meir et al. [16] and Beenakker [17] have developed a finite temperature transport theory for semiconductor quantum dots which can take into account Coulomb blockade effects and level discreteness. Meir et al. showed that the anomalous \( T \)-dependence of the peaks could be explained quite naturally if the zero temperature level widths of adjacent eigenstates fluctuate substantially (e.g. by a factor of four). Thus within their approach the anomalous \( T \)-dependence at higher temperatures is a direct consequence of the amplitude fluctuations at lower temperatures. Although their work accounted qualitatively for the temperature dependence, the level widths appear only as adjustable parameters in their theory with no microscopic model for their origin. A starting point for such a model is the expression for the maximum conductance (peak amplitude) in the single-level regime \( (kT \ll \Delta \epsilon) \) [17]

\[
g_{\text{max}} = \frac{e^2}{4\pi kT\hbar} \frac{\Gamma_\uparrow \Gamma_\downarrow}{(\Gamma_\uparrow + \Gamma_\downarrow)},
\]

where \( \Gamma_\uparrow \) and \( \Gamma_\downarrow \) are the partial decay widths into the left and right leads, and where \( (\Gamma_\uparrow + \Gamma_\downarrow) \) is the total decay width (henceforth denoted by \( \Gamma_\lambda \)) for level \( \lambda \). The second factor on the right-hand side in eq. (2) is simply the area under the \( T = 0 \) Breit–Wigner resonance. Eq. (2) is obtained by integrating this resonance over electronic energies weighted by the derivative of the Fermi function, which has the width \( kT \gg \Gamma_\lambda \); thus in this regime the resonance width is fixed to be \( \sim kT \) but its amplitude is controlled by the \( T = 0 \) decay widths (tunneling rates) from a given state. Hence the observed amplitude fluctuations are a measure of the fluctuations in these intrinsic widths and specifically of the \( T = 0 \) peak areas.

These intrinsic widths can be viewed as the product of two factors, the barrier penetration factor related to the height and shape of the potential in the classically-forbidden region, and a local density of states factor, relating to the spatial density of the quasi-forbidden region near the tunnel barrier. We assume that the dominant source for the peak amplitude fluctuations is the local density of states factor. This assumption is strongly supported by the data of fig. 2 which show that the peak amplitudes are sensitive to magnetic fields of order 10 mT. It is implausible that the barrier penetration factor would be sensitive to such weak fields, whereas the quasi-bound eigenstate can have such a high sensitivity due to the influence of the field on the phase of the state. A similar separation is made in the statistical theory of compound nuclear reactions between the reduced width which depends on the “resonance wave function”, and the barrier penetration factor which depends on the details
of the barrier and is monotonically increasing with energy [14]. It is only the reduced width which fluctuates between adjacent energy levels; the distribution of these fluctuations may be derived from a statistical ansatz for our case just as in the nuclear case.

6. R-matrix theory of reduced widths

In the standard approach to elastic scattering from nuclei [14] (for which spherical symmetry may be assumed) a linear relationship is derived in each angular momentum channel between the scattering wave function and its derivative at the interface between the nucleus and free space. The coefficient of proportionality is denoted by \( R(E) \); if \( M \) different decay channels exist this linear relationship defines a matrix of coefficients known as the \( R \) matrix for the nuclear reaction. In our case we do not have spherical symmetry so an angular momentum expansion is inappropriate, but we have a simplification due to the fact that only \( M \) propagating modes exist at the Fermi energy in each of the two leads. Moreover, in general the barrier penetration factor will be largest for tunneling into the lowest propagating mode, so we can in first approximation neglect all but this mode in each lead. With this approximation a derivation very similar so that used in the nuclear case yields [9] a \( 2 \times 2 \) \( R \) matrix of the form

\[
R_{mn}(E) = - \sum_{k=1}^{\infty} \frac{\gamma_{x_k}^m \gamma_{x_k}^n}{E - \varepsilon_k},
\]

(3)

where now the index \( m (= r, l) \) simply denotes the left or right leads, and the quantities \( \gamma_{x_k}^m \) and \( \varepsilon_k \) are determined by the solutions of the Schrödinger equation for the isolated quantum dot with appropriate boundary conditions. In particular \( \varepsilon_k \) are the energies of the quantum dot with Neumann boundary conditions imposed where the leads are attached and Dirichlet boundary conditions elsewhere [18]. If we denote the eigenfunction with energy \( \varepsilon_k \) by \( X_k(x, y) \) then we find

\[
\gamma_{x_k}^{1, r} = \sqrt{P_x} \int_{-W/2}^{W/2} dy \, \zeta(y) X_k(x, y),
\]

(4)

where \( P_x \) is the barrier penetration factor, \( x \) denotes the direction parallel to the lead and \( y \) that perpendicular, \( \zeta(y) \) is the transverse wave function in the leads (which have width \( W \)), and \( x_{1, r} \) is the position of the left (right) inner edge of the tunnel barriers. In deriving this expression we assume that the tunnel barrier is approximately uniform in the transverse direction; it should be straightforward to relax this assumption and we do not expect this to change the statistical results given below.

An exact non-linear relationship exists between the \( S \) matrix and the \( R \) matrix; however a particularly useful feature of this formulation is that as \( E \) approaches a particular \( \varepsilon_k \), the term in eq. (3) containing the corresponding \( X_k \) will dominate and all other terms may be neglected. In this “single-level” approximation the relationship between the \( S \) and the \( R \) matrices simplifies to yield the Breit–Wigner formula for the resonance line-shape under very general conditions. Moreover the level width which appears in this expression is simply

\[
\Gamma_{x_k}^{1, r} = \frac{\hbar^2 k}{m} |\gamma_{x_k}^{1, r}|^2 = \frac{\hbar^2 k}{m} P_x |\gamma_{x_k}^{1, r}|^2,
\]

(5)

where we have defined the reduced width \( \gamma_{x_k}^{1, r} \). In this discussion we will only treat the case of equal barriers on each side of the dot and hence \( P_x(E) \) will be the same on left and right. This means that the average decay widths to the left and right are equal, \( \bar{\Gamma} = \Gamma_{x_k}^{1, l} = \Gamma_{x_k}^{1, r} \) and we can define a dimensionless amplitude for the thermally-broadened resonance by

\[
\alpha_{x_k} = \frac{\Gamma_{x_k}^{1, r} \Gamma_{x_k}^{1, l}}{\bar{\Gamma}^2 + \Gamma_{x_k}^{1, r} \Gamma_{x_k}^{1, l}}.
\]

(6)

Returning now to our main goal, we can use eq. (6) to express the measured amplitude of the Coulomb blockade resonance arising from this state as \( g_{\text{max}} = (e^2/h)(\bar{\Gamma}/4\pi kT)\alpha_{x_k} \). Note that \( P_x(E) \) cancels in the expression for \( \alpha \), which only
depends on properties of the wave function within the barrier.

7. Distributions of amplitudes $\alpha$

In previous work [9] we have outlined the derivation of the probability density of the dimensionless amplitude variable, $\alpha$, from the ansatz that the ensemble of Hamiltonians for isolated quantum dots is described by the Gaussian orthogonal ensemble (GOE) in the presence of time-reversal (TR) invariance and by the Gaussian unitary ensemble (GUE) in its absence (i.e. for sufficiently large magnetic field) [6,19]. From eqs. (4) and (6) we see that the statistics of $\alpha$ will be determined by those of the wave functions $X_\alpha$. These statistics follow from our GOE (GUE) ansatz: if we expand $X_\alpha$ in terms of any $d$-dimensional regular basis the coefficients in that expansion are uniformly distributed on the surface of a $d$-dimensional unit sphere [19]. This statement combined with eqs. (4) and (6) allows us to show that the distribution of partial widths is $\chi^2$:

$$P(I_\alpha) = \chi^2(I_\alpha)$$

$$= \frac{\nu}{2(\nu/2 - 1)!I_\alpha} \left( \frac{\nu I_\alpha}{2I_\alpha} \right)^{(\nu/2-1)} \exp\left( \frac{\nu I_\alpha}{2I_\alpha} \right),$$

(7)

with $\nu = 1$ ($=2$) degrees of freedom for GOE (GUE). The case $\nu = 1$ is also known as the Porter–Thomas distribution in nuclear physics because the authors related it to the distribution.

![Analytic distributions of the normalized conductance amplitudes $\alpha$ obtained from the GOE (full curve) and GUE (dashed curve) ansatz of random matrix theory, eqs. (8) and (9). Also shown is a histogram of the distribution of $\alpha$ calculated using eqs. (4) and (6) for the Robnik dot without an external magnetic field.](image)
of compound nuclear reaction partial widths which it describes very well (note that time-reversal symmetry is never strongly broken in nuclei). If the dot (with the leads attached) has no parity symmetry then \( \Gamma'_{\pm} \) and \( \Gamma''_{\pm} \) are independently fluctuating variables and the total width, \( \Gamma_a \), is described by a \( \chi^2 \) distribution with twice the number of degrees of freedom. Knowledge of these distributions permits us to obtain the distribution \( \mathcal{P}_v(\alpha) \) from eq. (6) by integration over the component distributions. For the case \( v = 2 \) the integration can be performed completely analytically, and for the case \( v = 4 \) it may be reduced to a simple one-dimensional integral. One finds

\[
\mathcal{P}_2(\alpha) = \frac{2}{\sqrt{\pi \alpha}} e^{-2\alpha}, \tag{8}
\]

\[
\mathcal{P}_4(\alpha) = 2e^{-4\alpha} \int_0^\infty dz \ e^{-z} \sqrt{\frac{z + 4\alpha}{z}}. \tag{9}
\]

These distributions should describe the amplitude fluctuations of Coulomb blockade oscillations at zero magnetic field (\( v = 2 \)) and at a magnetic field sufficient to break fully time-reversal symmetry (\( v = 4 \)) as long as the classical mechanics in the dot is chaotic and parity symmetry is broken. They are universal in that they are insensitive to any further specific features of the geometry of the dot. In the event that the dot potential respects parity symmetry (which appears unlikely in the current experiments) then the relevant value of \( v \) is reduced by a factor of one half. It is trivial to show that \( \mathcal{P}_2(\alpha) \) is just the Porter–Thomas distribution. If the dot potential does not generate chaotic dynamics then the distribution of widths should be non-universal due to the presence of specific selection rules for tunneling transitions. We shall see below that in this case strong departures from eq. (7) are found numerically.

The distributions \( \mathcal{P}_2, \mathcal{P}_4 \) are plotted in fig. 4, where they are compared to numerical results to be described below. Note the substantial suppression of small amplitudes expected upon breaking TR symmetry; this leads to the simplest qualitative prediction of the theory: imposing a \( B \) field sufficient to break TR symmetry will measurably suppress the amplitude fluctuations. The behavior of \( \mathcal{P}_v(\alpha) \) as \( \alpha \to 0 \) can be understood by noting that for small amplitudes typically one of the partial widths \( \Gamma'_{\pm}, \Gamma''_{\pm} \) will dominate in \( \Gamma \) and hence will cancel in \( \alpha \), leading to a behavior of \( \mathcal{P}_v(\alpha) \) similar to \( \chi^2 \) as \( \alpha \to 0 \).

8. Effects of symmetry-breaking

Although in general the distribution \( \mathcal{P}_v \) cannot be expressed in closed form, all of its moments can be calculated analytically for arbitrary \( v \) [9] and some interesting results are predicted. First, breaking TR symmetry reduces amplitude fluctuations: the variances satisfy \( \Delta \alpha^2_{\pm} = 1/8 \approx 0.125 \), \( \Delta \alpha^2_{\pm} = 4/45 \approx 0.089 \). This follows simply from the decrease in the variance of the underlying \( \chi^2 \) distributions as \( v \) increases. Second, breaking TR symmetry increases the mean amplitude; \( \bar{\alpha}_{\pm} = 1/4, \bar{\alpha}_{\pm} = 1/3 \). Because it relates to \( \bar{\alpha} \), this prediction applies even when \( kT \gg \Delta \varepsilon \). A natural inference then is that such an effect should be observed in metallic systems and not just in semiconductor quantum dots. However, in metallic systems the eigenstates are typically coupled to thousands of channels in each lead (which makes the index \( v \) of the order 1000) and it is easy to show that the effect becomes negligible in this limit. The origin of the enhancement of the average amplitude can be seen qualitatively by expanding each partial width in \( \alpha \) for small variations around their mean which yields \( \bar{\alpha} \approx (1 - \Delta(\Gamma/\Gamma)^2)/4 \). Since \( \Delta(\Gamma/\Gamma)^2 \) decreases when TR symmetry is broken, \( \bar{\alpha} \) increases.

9. Breaking TR symmetry with a field

The previous predictions all assume that the magnitude of the magnetic field necessary to break time-reversal symmetry is sufficiently small that the field has negligible effect on the electron dynamics. The reason that such a weak field has any effect on the electronic states at all is of course through its coupling to the phase of the
wave function (essentially via the Aharonov–Bohm effect). Let us denote the field scale for TR symmetry-breaking by $B_c$. For the case of open mesoscopic systems for which conductance fluctuations have been extensively studied [20] it is now well-understood that $B_c^{\text{open}} \sim (\hbar/e)/A$ ($A$ being the sample area); the reason for this is that a diffusive trajectory crossing a sample encloses an area of order $A$, and if the sample is open it escapes as soon as it reaches one of the leads. For the open chaotic billiards used to model GaAs junctions $B_c$ is essentially the same order of magnitude, although its dependence on $A^{-1}$ is not strictly linear [21]. Study of the TR symmetry-breaking transition for closed chaotic systems (both ballistic and diffusive) has found that a parametrically smaller field scale controls the transition [6,13,22,23]. This smaller scale arises because for closed systems the electronic trajectories never escape and continue to accumulate phase from the field until a time of order $\hbar/\Delta e$ (after which semi-classical arguments do not apply). As the decay time for escape from the quantum dot is by assumption longer than $\hbar/\Delta e$ (since $\tilde{\Gamma} < \Delta e$), we should expect this field scale and not $B_c^{\text{open}}$ to control symmetry-breaking in our theory. For the case of chaotic billiards [6,13] this scale may be written as

$$B_c A = \Phi_e \sim \frac{\hbar}{e} \sqrt{\frac{\hbar v_f}{\Delta e \sqrt{A}}}.$$  

(10)

This $B_c$ should also set the scale for the sample-specific fluctuations of the amplitude pattern in a given sample shown in fig. 2; indeed an estimate of $B_c$ for this sample yields a value of the order 20 mT in rough agreement with the data. The dynamical effect of the field will certainly not be negligible when the electron cyclotron radius approaches the linear dimensions of the dot, $\sqrt{A}$; at this field scale, $B_0$, Landau quantization begins to play a significant role. Thus it is important that $B_0$ is much greater than $B_c$ in the experimental systems of interest. From the expression for $B_0$ and $B_c$ one finds that

$$\frac{B_0}{B_c} = C_0 N^{1/4},$$

(11)

where $N$ is the number of electrons on the dot and $C_0$ is a constant of order unity. Thus for typical experimental systems with $N \sim 100$ there should be a large interval of magnetic field over which our theory applies and may be tested by comparing to histograms of the data with and without a magnetic field.

10. Comparison to numerical results

We can test our predictions for $\mathcal{P}_\alpha(\alpha)$ by comparing them to histograms of $\alpha$ values generated directly from eigenfunctions of the Robnik model using eqs. (4) and (6). For the TR-invariant case the simple Robnik model described above is adequate as long as we restrict ourselves to eigenfunctions of a given parity (we chose positive parity arbitrarily). We imagine that the two leads are attached with angular width $\approx \pi/6$ at equally-spaced points along the upper half of the billiard (the interval $\theta \in [0, \pi]$) in all possible pairings. Restricting the leads to the upper half of the billiard avoids probing regions of the wave function related by parity, and we do not allow leads to overlap, so there should be no correlation between the wave-functions at each lead if the random-matrix ansatz is correct. For simplicity we neglect the curvature of the boundary within each lead. Values of $\tilde{\gamma}_\alpha$ are obtained simply by integrating each wave function over the appropriate angular interval weighted by the sine-wave transverse wave function of the lowest outgoing mode. This procedure then generates six partial widths for each wave function and hence fifteen uncorrelated values for $\alpha$ per level. Excellent statistics could then be obtained by considering just 30 levels. Levels between 120 and 150 were used which was sufficient to insure that the electron wavelength is short compared to the width of the leads, a necessary condition for the validity of the random-matrix ansatz. Initially we considered a shape parameter $p = 0.5$ corresponding to a strongly chaotic classical dynamics, so that our analytic theory should apply. Comparisons with our numerically generated results for the case $\nu = 2$ are given in fig. 4, where excellent agreement is found.
Comparisons of our analytic theory for the broken TR case ($\nu = 4$) is more involved. The parity symmetry of the Robnik billiards implies that even in the presence of a magnetic field the levels are still double-degenerate and described by the GOE not the GUE [6,13] (this "false" TR breaking in symmetric billiards is now well-understood in the quantum chaos literature). In order then to test the GUE predictions Berry and Robnik [13] introduced a cubic generalization of the Robnik billiard (often referred to as the "Africa" billiard because for some parameter values it is shaped like that continent) which breaks parity symmetry. They also showed how to include an Aharonov–Bohm flux in the conformal mapping of the Schrödinger equation and solve the generalized model with broken TR symmetry. We are currently implementing this model following the procedure outlined above to generate a histogram corresponding to $P(\alpha)$ shown in fig. 4. Although in our earlier work [9] we did generate such histograms (by direct fitting of the resonance spectrum to Breit–Wigner line shapes) it is relevant to perform the calculation suggested here for the following reason. In ref. [9] excellent agreement to random matrix theory was found in the absence of a magnetic field, but only mediocre agreement was found in the presence of a field. We attributed this discrepancy to the incipient effects of Landau quantization since we were only using the levels 25–50. This problem is avoided in the Robnik model since using an AB flux instead of a true uniform $B$ field isolates the phase effects without introducing any change in the dynamics [24].

A final interesting question which we can address within the Robnik model is the departure from universal statistics when the system is integrable or nearly integrable. In the chaotic regime, $p_i < p < p_c$, the distribution of e.g. partial widths should be unchanged as a function of shape and conform to eq. (7). For $0 < p < p_c$ as the system approaches integrability no such universality is expected. In fact we find (see fig. 5) that the distribution of partial widths changes continuously with shape in this regime, becoming broader as $p$ decreases. Moreover the distribution is no longer well-described by a $\chi^2_{\nu}$ even when we attempt to use $\nu$ as a fitting parameter. Thus our results indicate that the distribution of amplitudes should provide a sensitive probe of the properties of the dot potential which is able to distinguish nearly-integrable from chaotic potentials.

11. Effects of interactions

Before concluding we must briefly address the role of interactions in resonant tunneling through quantum dots, as the Coulomb blockade phenomenon which regulates the spacing of these resonances is manifestly an interaction effect. Both our numerical calculations and our discussion of $R$-matrix theory have neglected interactions. We note however that it is possible to formulate $R$-matrix theory more generally to take interactions into account formally [14]. When this is done the statistical predictions of random-matrix theory are not changed, although of course the particular level structure of a given system is changed by taking interactions into account. The fact that random-matrix theory was introduced to condensed matter physicists primarily through the theory of non-interacting disordered systems has tended to associate its domain of validity with the validity of non-interacting approximations. This is quite backwards, as the theory was invented precisely to describe strong interactions in resonant nuclear scattering, a system for which it is known to work extremely well [19]. Thus we expect our theory to apply quite well to the self-consistent energy levels of quantum dots at low magnetic field. Recently McEuen et al. [25] have shown that the behavior of the effective single-particle levels for quantum dots in high magnetic field is strongly altered by taking interactions into account self-consistently. However we note that the behavior found arises from the highly non-random character of the non-interacting wave functions in high field. In particular simply filling the non-interacting wave functions in the uppermost (spin-split) Landau levels compresses the electron gas far from its optimal density leading to an unphysical
increase in the electrostatic energy when the interactions are taken into account. We expect no such systematic effect at low field, and no qualitative difference between the self-consistent levels and non-interacting levels other than the average shifts due to the charging energy.

Another class of interaction effects which has been proposed as relevant to quantum dots is a Kondo-like pinning of the resonance to the Fermi level [26–28]. This effect arises in principle from complicated correlations between the wavefunction on the dot and the Fermi sea of the leads; no analogue to it exists in the nuclear case where the asymptotic state of the system (projectile far from target) is non-interacting. This type of effect in condensed matter systems would then limit the range of validity of random-matrix theory (and of our particular application of it). We note however that this effect appears to require $kT \ll \bar{\Gamma}$ [26], thus it is not relevant in the thermally-broadened regime we have treated.

12. Summary

In summary, we have shown that the energies and wave functions in two-dimensional quantum dots are extremely sensitive to the shape of the dot. This observation calls into question the feasibility of making truly periodic quantum dot superlattices. In addition we have argued that amplitude fluctuations of Coulomb blockade
oscillations in semiconductor quantum dots arise from fluctuations in the decay widths of the quasi-bound states. If the classical dynamics in the dot potential is fully chaotic these amplitude fluctuations will have universal probability distributions which we have derived from random-matrix theory. For these distributions a weak magnetic field enhances the mean amplitude while reducing fluctuations in the amplitude, an effect which should be experimentally observable. In addition we find non-universal distributions as the system approaches integrability. Hence this amplitude distribution is extremely sensitive to many features of the dot potential and may provide an experimental probe of its basic features.

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References

[18] The choice of Neumann boundary conditions at the leads is for convenience and conforms to the standard $R$-matrix literature [14]. This choice has no physical significance and a different choice would lead to a different expression for the $R$ matrix, a different relationship between the $R$ and $S$ matrices, but of course the same $S$ matrix. In contrast the use of Dirichlet boundary conditions at the boundaries of the dot away from the leads is necessary to formulate the $R$-matrix theory, which depends on current conservation to relate quantities within the dot to the current at the leads. If Dirichlet boundary conditions are not used then current flows "through the walls of the dot" and it appears that no sensible $R$-matrix theory can be defined. Furthermore, it is well known that Dirichlet boundary conditions may not be used at the leads [14]; hence the wave function appearing in the $R$-matrix theory for quantum dots should satisfy mixed boundary conditions on its surface. This presents a significant obstacle to evaluating the $S$ matrix for quantum dots from a finite approximation to the $R$ matrix, as eigenfunctions satisfying mixed boundary conditions cannot be obtained straightforwardly by the Robnik technique used for the calculations of fig. 3 (for which Dirichlet boundary conditions were used). This practical limitation does not exist in the spherically-symmetric case of nuclei. However, it is plausible to assume that the statistical quantities such as the distribution of level widths are independent of the boundary conditions. We make this assumption in our comparisons of eq. (8) to the results from the Robnik model with uniform boundary conditions and find excellent agreement (see fig. 4).