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Mechanism of plateau formation in the quantum Hall effect

H. Bruus[1], O.P. Hansen[2], and E.B. Hansen[2]

[1] Danish Institute of Fundamental Metrology, Lyngby, Denmark

[2] Physics Laboratory, University of Copenhagen, H.C. Ørsted Institute, Copenhagen, Denmark

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As the magnetic flux is increased (decreased) relative to the mid-plateau value by a given number of flux quanta, the 2-dimensional electron liquid responds by forming the same number of vortices (antivortices). Assuming the vortices (antivortices) to be pinned, we show that with a transport current flowing by, the pinning centres exert a force on the electron liquid. Inclusion of this force in a force balance equation for the electron liquid explains the formation of plateaus within the FQHE. The possibility of generalizing the theory to the IQHE is discussed.

Introduction

The integer quantum Hall effect and the fractional quantum Hall effect are usually thought of as rather distinct effects. Whereas electron-electron interaction is considered fundamental in FQHE, the IQHE, in its essence, is regarded as a single particle localization phenomenon. A comprehensive up-to-date theoretical and experimental review of the QHE is given in [1].

The formation of plateaus is considered to be well understood within the IQHE. Although this is not the case in the FQHE, it is believed that an explanation can be found through analogy with the IQHE. In the present paper we turn things around. We propose an explanation of plateau formation in FQHE and then discuss the possibility of extending this explanation to the IQHE. A preliminary version of our explanation was given in [2]. In this paper we give a more thorough presentation of our theory. Most important is that we have been able to prove the expression for the force from the pinning centres, which in [2] was introduced as an assumption.

Our explanation of plateau formation in the FQHE is based on three elements: *(I)* The vortex picture. *(II)* The force from pinning centres. *(III)* The principle of force balance.

(I) The essentials of the vortex picture at $T=0$ are the following: at the filling factor $\nu=\frac{1}{m}$, m being an odd integer, we have the mid-plateau state described by Laughlin's ground state wavefunction [3], which is without vorticity. However, as the magnetic flux is increased (decreased) relative to the mid-plateau value by a given number of flux quanta, the electron liquid responds by forming the same number of fractionally charged vortices (antivortices) [4, 5]. The spatial structure and the velocity field of a vortex (an antivortex) are qualitatively described by Laughlin's quasi-hole (quasi-electron) wavefunction [3]. A vortex and an antivortex have charges of opposite signs and, closely related to this, opposite directions of rotation. It is a remarkable feature of the vortex picture that in the region between the vortices (antivortices) the electron density is changed as a function of the magnetic flux in such a way as to maintain the filling factor at the canonical value $\nu=\frac{1}{m}$. The vortices (antivortices) are expected to remain pinned in the presence of a transport current that does not exceed the critical current where the FQHE breaks down, thus in this respect our explanation supports the view that plateau formation in the FQHE is due to localization of quasi-particles [3, 6, 7].

(II) Let us consider a pinned vortex (antivortex) with a transport current flowing by. The interplay between the rotational motion of the vortex (antivortex) and the linear motion of the transport current produces an interesting phenomenon: the pinning centre exerts a force on the electron liquid. It will

be shown that in the case of a pinned vortex this force has the form $\mathbf{f}_P = \boldsymbol{\Phi}_0 \times \mathbf{j}$, where $\mathbf{j}$ is the two-dimensional current density, $|\boldsymbol{\Phi}_0| = \frac{h}{e}$, and where $\boldsymbol{\Phi}_0$ is parallel to the magnetic field. For a pinned antivortex the sign is opposite: $\mathbf{f}_P = -\boldsymbol{\Phi}_0 \times \mathbf{j}$. Knowing the number of pinned vortices (antivortices), we obtain an expression for the total force, $\mathbf{F}_P$, which the pinning centres exert on the electron liquid.

(III) In the QHE-experimental set-up, where the transverse component of the current is zero, the total transverse force on the electron system must also be zero:

$$\mathbf{o} = \mathbf{F}_P + \mathbf{F}_M + \mathbf{F}_E \tag{1}$$

where $\mathbf{F}_M$ is the magnetic force on the electron liquid and $\mathbf{F}_E$ is the electric force.

It will be shown that the force balance condition (1) implies that the Hall response, V_H, is independent of the magnetic field in an interval around $B_{\nu=\frac{1}{m}}$ and that $V_H = m\frac{h}{e^2}I$ in this interval. Finally, the proof is extended to the case $\nu = \frac{p}{q}$, where q is an odd integer.

The vortex picture of the FQHE

In this section we first discuss the mid-plateau states, which occur at the magnetic fluxes $\Phi = mN\Phi_0$, N being the number of electrons, then the vortex states, which arise as the magnetic flux is increased relative to a mid-plateau value, and finally the antivortex states, which are generated as the magnetic flux is decreased relative to a mid-plateau value. The states of the electron system between Hall plateaus are not discussed. A schematic illustration of two neighbouring Hall plateaus is given in Fig. 1.

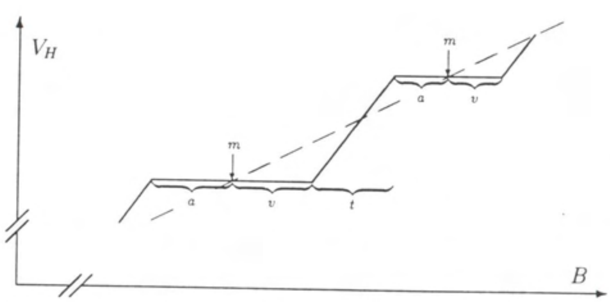

Fig. 1. The states in the FQHE. The mid-plateau states are marked by "m". The vortex states exist in the invervals "v", and the antivortex states in the intervals "a". The transition regions are designated by "t"

Let us first consider the case where the magnetic flux, Φ, has a mid-plateau value, i.e. $\Phi = mN\Phi_0$. In this case the ground state of the electron liquid [3] is described by

$$\Psi_{\nu=\frac{1}{m}} = \prod_{i>k}^{N} (z_i - z_k)^m \exp\left\{-\sum_{n=1}^{N} \frac{|z_n|^2}{4l^2}\right\} \tag{2}$$

where l is the magnetic length, and $z = x - iy$ (corresponding to an arrangement where the magnetic field points in the z-direction).

We imagine that all electrons, except electron 1, are kept at arbitrarily fixed positions [8]. It follows then from the structure of (2) that the zeros of $\Psi_{\nu=\frac{1}{m}}(z_1)$ are attached to the fixed particles, each particle yielding an m-double zero. The velocity field, $\mathbf{v}_1$, of electron 1 is

$$\mathbf{v}_1 = \frac{\mathbf{S}_1}{\Psi^*_{\nu=\frac{1}{m}} \Psi_{\nu=\frac{1}{m}}} \tag{3}$$

where

$$\mathbf{S}_1 = \frac{\hbar}{M}\frac{1}{2i}\left[\Psi^*_{\nu=\frac{1}{m}} \nabla_1 \Psi_{\nu=\frac{1}{m}} - \Psi_{\nu=\frac{1}{m}} \nabla_1 \Psi^*_{\nu=\frac{1}{m}}\right] + \frac{e}{M}\mathbf{A}\Psi_{\nu=\frac{1}{m}} \Psi^*_{\nu=\frac{1}{m}} \tag{4}$$

is the probability current density. By use of (2), (3), and (4), see Appendix A, we find that

$$\mathbf{v}_1 = \frac{m\hbar}{M}\sum_{j=2}^{N}\left(+\mathrm{Im}\left\{\frac{1}{z_1 - z_j}\right\},\ -\mathrm{Re}\left\{\frac{1}{z_1 - z_j}\right\}\right) + \frac{e}{M}\mathbf{A}. \tag{5}$$

We calculate the line integral of $\mathbf{v}_1$ along a circle centred at the origin and with radius r:

$$\oint \mathbf{v}_1 \cdot d\mathbf{s}_1 = \frac{h}{M}\left(-mN_r + \frac{\Phi_r}{\Phi_0}\right) \tag{6}$$

where N_r is the number of electrons with positions inside the circle, and Φ_r is the flux enclosed by the circle. The circulation of the velocity contains a paramagnetic contribution, governed by the number, mN_r, of zeros, and a diamagnetic contribution determined by the number of flux quanta within the circle.

The average number of electrons, $\langle N_r \rangle$, within the circle is

$$\langle N_r \rangle = \frac{1}{m}\frac{\Phi_r}{\Phi_0}. \tag{7}$$

The validity of (7) follows from the fact the ground state (2) has a uniform density. Substitution of (7) into (6) then shows that

$$\langle \oint \mathbf{v}_1 \cdot d\mathbf{s}_1 \rangle = 0. \tag{8}$$

If (8) were not fulfilled, large "hydrodynamical" energies would come into play [8]. In Laughlin's wavefunction $\Psi_{\nu=\frac{1}{m}}(z_1)$ there are m zeros directly on each particle. In the true ground state, it seems, $m-1$ of these zeros move a little away from the particle [4, 9], but this will not change the validity of (8).

The mid-plateau state is unique because everywhere, on the average, the paramagnetic and the diamagnetic contribution to the velocity field exactly cancel each other. On the average there is a perfect balance, within a given area, between the number of zeros and the number of flux quanta.

Then consider the situation where the total flux has been increased by one flux quantum as compared to the mid-plateau value $mN\Phi_0$. In this case, as discussed in [5], the ground state can be qualitatively described by

$$\Psi_h = \prod_{j=1}^{N} z_j \prod_{i>k}^{N} (z_i - z_k)^m \exp\left\{-\sum_{n=1}^{N} \frac{|z_n|^2}{4l^2}\right\} \tag{9}$$

which, except for a minute change in the magnetic length, is identical to Laughlin's quasi-hole excitation at $\nu = \frac{1}{m}$. Calculating the circulation of $\mathbf{v}_1$ for (9), see Appendix A, and taking average values we obtain

$$\langle \oint \mathbf{v}_1 \cdot d\mathbf{s}_1 \rangle = \frac{h}{M}\left(-1 - m\langle N_r^+ \rangle + \frac{\Phi_r^+}{\Phi_0}\right) \tag{10}$$

where $\langle N_r^+ \rangle$ is the average number of electrons inside the circle, and where Φ_r^+ is the flux enclosed by the circle. The contribution, -1, in the bracket on the right hand side in (10) arises because $\Psi_h(z_1)$, besides the zeros of the ground state, has an extra zero at the origin. In the limit $r \ll l$ we obtain

$$\langle \oint \mathbf{v}_1 \cdot d\mathbf{s}_1 \rangle = -\frac{h}{M} \tag{11}$$

displaying the clockwise rotation of the liquid in the vortex region. The magnetic field created by the vortex current can be shown to be negligibly small as compared to the external field (see Appendix B). Taking the radius, r, to be a few magnetic lengths, we thus have that Φ_r^+ is essentially equal to Φ_r, while, according to Laughlin [3] the vortex is fractionally charged,

$$\langle N_r^+ \rangle = \langle N_r \rangle - \frac{1}{m},$$

showing that $\langle \oint \mathbf{v}_1 \cdot d\mathbf{s}_1 \rangle = 0$.

Finally we consider what happens to the electron liquid when the flux is reduced by one flux quantum. Qualitatively the situation can be described by Laughlin's quasi-electron wavefunction with the appropriate magnetic length

$$\Psi_p = \left[\prod_{j=1}^{N} \frac{\partial}{\partial z_j} \prod_{i>k}^{N} (z_i - z_k)^m\right] \exp\left\{-\sum_{n=1}^{N} \frac{|z_n|^2}{4l^2}\right\}. \tag{12}$$

This wavefunction also describes a rotating structure, an antivortex, but now with a local surplus of $\frac{1}{m}$ of an electron charge in the defect region. If follows from (12) that the antivortex rotates counter-clockwise.

The charge densities of a vortex and an antivortex at $\nu = \frac{1}{3}$ are displayed in [10, 11]. Qualitative features of a vortex can be elucidated by studying (9) for $m = 1$. In this case Laughlin's ground state wavefunction is identical with the Slater determinant generated from all the single particle states of the lowest Landau level. This simplifies the vortex wavefunction (9) to such an extent that it is possible to obtain analytical expressions for the velocity field and the electron density:

$$\begin{aligned} \mathbf{v}(r, \theta) &= -\frac{h}{M}\frac{1}{2\pi r}\exp\left\{-\frac{\pi r^2}{a_0}\right\}\mathbf{e}_\theta \\ n(r, \theta) &= \frac{1}{a_0}\left[1 - \exp\left\{-\frac{\pi r^2}{a_0}\right\}\right] \end{aligned} \tag{13}$$

where $a_0 = 2\pi l^2 = \frac{h}{eB}$. A derivation of (13) is given in Appendix B. It is seen from (13) that a vortex effectively can be depicted as an area $2\pi l^2$ completely depleted of electrons.

By a generalization of the preceding discussion we arrive at the vortex picture: the mid-plateau states are outstanding by being without vorticity. However, as the flux is increased (decreased) relative to a mid-plateau value by a given number of flux quanta, the electron liquid responds by forming the same number of vortices (antivortices) [4, 5]. These vortices (antivortices), we expect, will be uniformly distributed across the sample. Thus the density of vortices, n_v, and the density of antivortices, n_a, are

$$\begin{aligned} n_v &= \frac{eB}{h} - mn \quad \text{for } B > B_{\nu=\frac{1}{m}} \\ n_a &= mn - \frac{eB}{h} \quad \text{for } B < B_{\nu=\frac{1}{m}}. \end{aligned} \tag{14}$$

At a mid-plateau state we have the constant electron density $n_0 = \frac{1}{m}\frac{eB_0}{h}$, where B_0 is short for $B_{\nu=\frac{1}{m}}$. In a vortex state each vortex has a deficit of $\frac{1}{m}$ of an electron charge [3]. The deficit charge is pushed out into the area between the vortices, and it is precisely because of this feature that the electron density in the inter-vortex area is increased to the value $n = \frac{1}{m}\frac{eB}{h}$, thereby maintaining the canonical filling factor $\nu = \frac{1}{m}$ in this region. To clarify the picture we emphasize again that the creation of a vortex is effectively equivalent to the formation, in the electron liquid, of an area $2\pi l^2$ completely depleted of charge. Conversely, in an antivortex state, charge is removed from the area between the antivortices so that the canonical filling factor $\nu = \frac{1}{m}$ can be maintained in this region, and $\frac{1}{m}$ of an electron charge thus can be supplied to each antivortex (see Fig. 2). The creation of an antivortex is effectively equivalent to the superimposition on the otherwise homogeneous electron liquid of $\frac{1}{m}$ of an electron charge within an area $2\pi l^2$.

As more and more vortices (antivortices) are created in the liquid the order in terms of the $\nu = \frac{1}{m}$-correlation gradually breaks down. Initially it is energetically favourable for the electron system to establish the canonical particle density, $\frac{eB}{mh}$, in so large an area as possible. Therefore it yields only locally, by formation of vortices (antivortices), to the increase (decrease) of the magnetic field. However, the gain in correlation energy outmatches the disadvantage of the resulting inhomogeneity. But as B moves further away from B_0 the latter disadvantage increases, and it becomes increasingly difficult to maintain the favourable $\nu = \frac{1}{m}$-correlations. Therefore (14) holds only in a certain interval around B_0.

Recently promising progress has been made as regards the establishment of a Ginzburg-Landau theory of the FQHE [4]. A more comprehensive description of the vortex state will probably require a G.L.-theory.

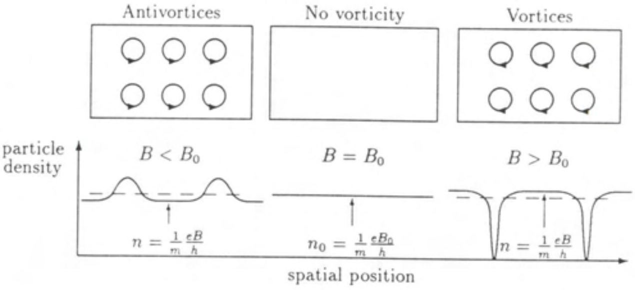

Fig. 2. Vortex states in FQHE sketched at three different magnetic fields: below, at, and above the mid-plateau value $B_0 = B_{\nu=\frac{1}{m}}$. The top part of the figure shows the vortices and antivortices in the sample and their directions of rotations for a magnetic field pointing out of the paper plane. The lower part shows the particle density as a function of the spatial position. Note that $\nu = \frac{1}{m}$ in the vortex/antivortex free region. The shapes, especially the shape of the antivortex, should not be taken literally

The force from the pinning centres

In the following we discuss the situation where the magnetic field is larger than the mid-plateau field, but where the density of vortices is not so large that they perturb each other appreciably. Now consider one particular of these vortices. This vortex, we imagine, is pinned, but to a "pin" that we can move around (see Fig. 3). Initially the "pin" is located at some position a. Let $\mathbf{f}_P$ denote the force which the pinning centre exerts on the electron liquid flowing by. It is this force which we must supply when, quasi-statically (carefully avoiding other vortices) we move the "pin", and thus the surrounding vortex, from a to another position b.
In doing this we perform the work

$$W = \int_a^b \mathbf{f}_P \cdot d\mathbf{s}. \tag{15}$$

This work must be equal to the increase of energy. However, since the overall result of the quasi-static

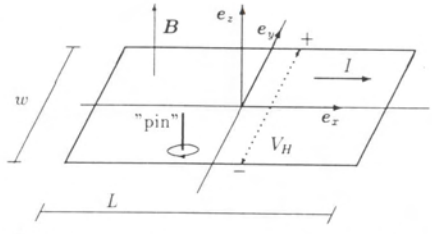

Fig. 3. The geometry of the system. The description uses the coordinate system $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. The magnetic field, **B**, points in the z-direction. A transport current, I, is passed through the sample in the x-direction, thus giving rise to a Hall field in the negative y-direction. The length and the width of the sample are denoted L and w. A vortex bound to a "pin" is shown

transport is the displacement of the charge $-\frac{e}{m}$ from b to a (equivalent to the displacement of the positively charged vortex from a to b), we have

$$W=-\frac{e}{m}\int_a^b \mathbf{E}\cdot d\mathbf{s}. \tag{16}$$

Since (15) and (16) are valid for arbitrary a and b we conclude that

$$\mathbf{f}_P=-\frac{e}{m}\mathbf{E}. \tag{17}$$

However, the electrical field, $\mathbf{E}$, can be related to the current density that would exist at the position of the vortex if the vortex were not there[1]. Noting that in the region between vortices the filling factor is maintained at its canonical value $\nu=\frac{1}{m}$ (see Fig. 2), we conclude that the current density between the vortices is given by the local quantum Hall relation

$$j=\frac{1}{m}\frac{e^2}{h}|\mathbf{E}|. \tag{18}$$

The geometry of Fig. 3 and (18) implies $-\frac{e}{m}\mathbf{E}=\boldsymbol{\Phi}_0\times\mathbf{j}$, and consequently the validity of

$$\mathbf{f}_P=\boldsymbol{\Phi}_0\times\mathbf{j}. \tag{19}$$

The pinning force for an antivortex, $\mathbf{f}_P=-\boldsymbol{\Phi}_0\times\mathbf{j}$, can be derived by the same line of reasoning.

The case where the density of vortices is so large that the individual vortex to a considerable extent experiences the velocity field of other vortices is quite complicated, and we shall not discuss it here.

Let us consider a vortex state. Then, according to (14) and (19), the force on the electron liquid from the pinning centres is

$$\mathbf{F}_P=L\int n_v\,\mathbf{f}_P\,dy=L\left(\frac{eB}{h}-mn\right)\Phi_0\int|\mathbf{j}(y)|\,dy\,\mathbf{e}_y$$
$$=L\left(\frac{eB}{h}-mn\right)I\Phi_0\,\mathbf{e}_y \tag{20}$$

implying that

$$\mathbf{F}_P=\left(\frac{\Phi}{\Phi_0}-mN\right)\Phi_0\frac{I}{w}\mathbf{e}_y. \tag{21}$$

[1] It is the current density which exists on both sides of the vortex a few magnetic lenghts, l, away from the centre of the vortex

According to (14) the expression for n_a can formally be obtained from the expression for n_v by a change of sign. Since the pinning force for an antivortex is given by (19) except for a change of sign, it is seen that the expression (21) holds also for an antivortex state.

The magnetic and the electric force

The number, N, of electrons is assumed to be uninfluenced by the presence of a transport current. We use the geometry shown in Fig. 3. In order to simplify the discussion we consider the limits where $w\gg l$ and $L\gg w$, thereby avoiding a discussion of end effects.

We start by considering the magnetic force which is described by the operator

$$\hat{\mathbf{F}}_M=-e\sum_{i=1}^N\hat{\mathbf{v}}_i\times\mathbf{B}. \tag{22}$$

Thus the magnetic force takes the form

$$\mathbf{F}_M=\langle\hat{\mathbf{F}}_M\rangle=-e\sum_{i=1}^N\langle\hat{\mathbf{v}}_i\rangle\times\mathbf{B}=-BIL\mathbf{e}_y \tag{23}$$

where $\langle\ldots\rangle$ means the quantum mechanical expectation value, and where we have used

$$I=\left|\frac{-e}{L}\sum_{i=1}^N\langle\hat{\mathbf{v}}_i\rangle\right|.$$

Let $U(\mathbf{r})$ denote the electric potential at the position $\mathbf{r}$. In dealing with the electric force, $\mathbf{F}_E=\left\langle -e\sum_{i=1}^N(-\nabla_{\mathbf{r}}U(\mathbf{r}_i))\right\rangle$, we first consider the case where the electric field is constant across the sample. In this case we find

$$\mathbf{F}_E=\left\langle -e\sum_{i=1}^N\mathbf{E}(\mathbf{r}_i)\right\rangle=eN|\mathbf{E}|\,\mathbf{e}_y=eLn\,V_H\,\mathbf{e}_y \tag{24}$$

where $V_H=w|\mathbf{E}|$ is the Hall response, and where and $n=\frac{N}{Lw}$ is the average electron density. Then let us consider the general case, where the electric field varies across the sample. Experiments [12, 13] show that the electric field varies slowly with l as scale. Therefore we can divide the sample in strips of the width $w_j\gg l$, containing N_j electrons, and across which the electric field has a constant value $\mathbf{E}_j$. Let $n_j\equiv\frac{N_j}{Lw_j}\equiv n+\delta n_j$

denote the average electron density in the j-th strip. The electric force can then be written as

$$\mathbf{F}_E=\sum_j eLw_j n_j|\mathbf{E}_j|\,\mathbf{e}_y=\sum_j eLn_j V_H^j\,\mathbf{e}_y \tag{25}$$

where $V_H^j\equiv|\mathbf{E}_j|\,w_j$ is the voltage drop across the j-th strip. Thus we obtain

$$\mathbf{F}_E=eLn\left(\sum_j V_H^j+\sum_j|\mathbf{E}_j|\frac{w_j\,\delta n_j}{n}\right)\mathbf{e}_y. \tag{26}$$

An estimation based on the measured curvature of the potential distribution [13] indicates that

$$\frac{|\delta n_j|}{n}\ll 1. \tag{27}$$

Furthermore, since the condition $\sum_j w_j\,\delta n_j=0$ tends to reduce the second term on the right-hand side in (26), we conclude that this term is minute compared to the first. Although this minute term must be taken into account when the ultimate accuracy of the QHE is to be decided, we neglect it for the time being. Thus, since $V_H=\sum_j V_H^j$, we find that also in the general case the electric force is given by (24).

Plateau formation by force balance

By substitution of (21), (23), and (24) into (1) we obtain

$$BIL=LneV_H+L\left(\frac{eB}{h}-mn\right)\Phi_0\,I \tag{28}$$

which implies

$$V_H=m\frac{\Phi_0}{e}I=m\frac{h}{e^2}I \tag{29}$$

showing that the Hall response V_H comes out independent of B, which is tantamount to the formation of a plateau.

Let us now consider the plateau around some filling factor $\nu_0=\frac{p}{q}$, q being an odd integer. According to the hierarchical theories [14–16] quasi-particles (i.e. vortices and antivortices) will have the charges $\pm\frac{e}{q}$. Therefore it will be possible to maintain the canonical filling factor $\nu_0=\frac{p}{q}$ in the region between vortices (antivortices) only by forming p vortices (antivortices) for each surplus (deficit) flux quantum. Thus the densities of vortices and antivortices are given by

$$\begin{aligned} n_v&=p\left(\frac{eB}{h}-\frac{1}{\nu_0}n\right) \quad \text{for } B>B_{\nu=\nu_0}\\ n_a&=p\left(\frac{1}{\nu_0}n-\frac{eB}{h}\right) \quad \text{for } B<B_{\nu=\nu_0}. \end{aligned} \tag{30}$$

However, if the vortex charge is $+\frac{e}{q}$, the expressions (17) and (18) must be replaced by the expressions: $\mathbf{f}_P=-\frac{e}{q}\mathbf{E}$ and $j=\frac{p}{q}\frac{e^2}{h}|\mathbf{E}|$, implying

$$\mathbf{f}_P=\frac{1}{p}\boldsymbol{\Phi}_0\times\mathbf{j} \tag{31}$$

which then together with (30) leads to the following expression for the force from the pinning centres

$$\mathbf{F}_P=\left(\frac{\Phi}{\Phi_0}-\frac{q}{p}N\right)\Phi_0\frac{I}{w}\mathbf{e}_y. \tag{32}$$

By the same argument as applied earlier we find that (32) is valid also for antivortex states. Substitution of (32) into the force balance equation (1) leads to a Hall plateau with $V_H=\frac{q}{p}\frac{h}{e^2}I$.

Single particle localization and localization of quasi-particles

It has been suggested [3, 6, 7] that the localization of quasi-particles in the FQHE can be considered as being analogous to the single particle localization of the IQHE. We argue that the two concepts are different.

In the heterostructure, usually, the magnetic field is the external control variable, whereby we can regulate the filling factor. In the MOSFET the filling factor is regulated by changing the electron density n, via the gate voltage. In the following we shall think of n as the control variable.

Let us compare the plateaus around $\nu=1$ and $\nu=\frac{1}{3}$ at $T=0$. Whereas, according to the vortex picture, the mid-plateau state at $\nu=\frac{1}{3}$ is without localization, the single particle description of the IQHE insists that localization is present, i.e. the localized states in the tails of $\nu=1$-band are occupied [17–19].

Then imagine that n is reduced. In the single particle picture this leads to a *decrease* of localization in the sense that localized states, which were formerly occupied, are now gradually being emptied. In the

$\nu=\frac{1}{3}$ case, the reduction of n causes the formation of quasi-particles (vortices), which subsequently becomes localized (pinned), i.e. in this case there is an *increase* of localization.

In the single particle theory of the IQHE, localization is an affair between the electrons and the impurities. In contradistinction, localization in the FQHE is due to electron-electron interaction in the sense that the objects, which (via their interaction with the impurities) can become localized, owe their very existence to the electron-electron interaction.

Discussion

Suppose, starting at $\nu=1$, that we increase the magnetic field. How will the electron system respond? We speculate that it will respond by forming vortices described by the wavefunction $\Phi_\nu=\prod_i z_i\,\Phi_{\nu=1}$, where $\Phi_{\nu=1}$ is the true ground state at the filling factor $\nu=1$. Correspondingly, we imagine that a decrease of the magnetic flux causes the production of antivortices, so that also around $\nu=1$ we have a situation of the kind indicated in Fig. 1.

Comparing Landau level splittings and typical electron-electron interaction energies for $n=3\cdot 10^{15}\ \mathrm{m}^{-2}$ and $\nu=1$ we find

$$
\begin{aligned}
&\text{Si/SiO}_2\text{ interface:} && \frac{1}{4\pi\varepsilon_0\,\varepsilon_r}\frac{e^2}{l}=17\ \mathrm{meV}\\
& && \hbar\omega_c=3\ \mathrm{meV}\\
&\text{GaAs/GaAlAs interface:} && \frac{1}{4\pi\varepsilon_0\,\varepsilon_r}\frac{e^2}{l}=16\ \mathrm{meV}\\
& && \hbar\omega_c=22\ \mathrm{meV}.
\end{aligned}
\tag{33}
$$

The conditions stated in (33) suggest that $\Phi_{\nu=1}$ has, to considerable extent, components after higher Landau levels. Thus $\Phi_{\nu=1}$ describes a more correlated motion of the electrons than $\Psi_{\nu=1}$ given in $(B1)$, meaning that $\langle\Phi_{\nu=1}|\frac{1}{r_{ij}}|\Phi_{\nu=1}\rangle$ is smaller than $\langle\Psi_{\nu=1}|\frac{1}{r_{ij}}|\Psi_{\nu=1}\rangle$. Although $\Phi_{\nu=1}$ may thus be rather different form $\Psi_{\nu=1}$, it nevertheless describes a state with a uniform electron density. As a consequence the velocity field and the density profile of Φ_ν is determined essentially by the product part, $\prod_i z_i$, of the wavefunction. Therefore, the velocity field and density profile of Φ_ν will resemble (13).

A few years back experiments had established the existence of plateaus at integer filling factors [20, 21] and at $\nu=\frac{1}{3}$ [22]. Today we have continuous curves of ρ_{xx} and ρ_{xy} versus B ranging from $\nu=\frac{2}{5}$ to $\nu>4$ [23, 24]. For $\nu<1$ an impressive amount of structure, roughly symmetric around $B_{\nu=\frac{1}{2}}$, has been uncovered. But also in the intervals $1<\nu<2$ and $2<\nu<3$ there is a lot of structure. Inspecting Fig. 1 in [23], which shows integer and fractional plateaus side by side (for example the $\nu=1$-plateau has fractional plateaus on both sides), we find it difficult to imagine that the integer and fractional plateaus are formed by different mechanisms.

The hypothesis stated above that the vortex picture applies also to the IQHE, thereby making the vorticity a kind of hallmark of the quantum Hall effect, would allow a common explanation of all plateau formation, implying that we are dealing, basically, with one effect.

We want to thank Jørgen Holst for helpful discussions.

Appendix A

Considering the situation where all electrons, except electron 1, are kept at arbitrary fixed positions, we calculate the velocity field of electron 1 for the ground state (2) and the vortex state (9).

Noting that $z_1=x_1-iy_1$ (corresponding to the fact that **B** points in the z-direction), we have

$$
\begin{aligned}
\nabla_1(z_k-z_1)^m&=(\partial_{x_1}(z_k-z_1)^m,\ \partial_{y_1}(z_k-z_1)^m)\\
&=\frac{m}{z_k-z_1}(-1,\ +i)(z_k-z_1)^m
\end{aligned}
\tag{A 1}
$$

by the use of which we obtain

$$
\begin{aligned}
\nabla_1\,\Psi_{\nu=\frac{1}{m}}=&\left[\left\{\sum_{k=2}^{N}\frac{m}{z_1-z_k}\right\}(+1,\ -i)\right.\\
&\left.+\left(\frac{-x_1}{2l^2},\ \frac{-y_1}{2l^2}\right)\right]\Psi_{\nu=\frac{1}{m}}.
\end{aligned}
\tag{A 2}
$$

Substitution of (A 2) into (4) and use of (3) leads to the expression (5) for the velocity field.

In order to investigate the vortex properties of the velocity field it is desirable to calculate the circulation $\oint \mathbf{v}_1\cdot d\mathbf{s}_1$ along a circle, Γ, centered at the origin and with radius r. The diamagnetic contribution to the line integral is $\frac{e}{M}\Phi_r=\frac{h}{M}\frac{\Phi_r}{\Phi_0}$, where Φ_r is the flux enclosed by the circle. To find the paramagnetic contribution we use the calculus of residues, and from (5) it is seen that the complex function

$$f(z)=\sum_{j=2}^{N}\frac{m}{z-z_j} \tag{A 3}$$

comes into play. In order to use standard calculus of residues we rename the complex variable $z=x-iy$ occurring in (A 3): $z\to z^*=x+iy$, i.e. we complex conjugate all our expressions. This changes the paramagnetic contribution to the velocity field (5) in the following way:

$$\mathbf{v}_{\text{para}}\equiv\frac{\hbar}{M}(+\operatorname{Im}f(z),\,-\operatorname{Re}f(z))$$
$$\to\mathbf{v}_{\text{para}}=\frac{\hbar}{M}(-\operatorname{Im}f(z),\,-\operatorname{Re}f(z)). \tag{A 4}$$

The line integral thus takes the form

$$\begin{aligned}\oint\mathbf{v}_{\text{para}}\cdot d\mathbf{s}&=\frac{\hbar}{M}\oint[-\operatorname{Im}f(z)\,dx-\operatorname{Re}f(z)\,dy]\\&=\frac{\hbar}{M}\operatorname{Re}\{\oint[(-\operatorname{Im}f(z))\,dx-(\operatorname{Re}f(z))\,dy]\\&\quad+i[(\operatorname{Re}f(z))\,dx+(-\operatorname{Im}f(z))\,dy]\}\\&=\frac{\hbar}{M}\operatorname{Re}\{\oint_\Gamma[-\operatorname{Im}f(z)+i\operatorname{Re}f(z)]\,dz\}\\&=\frac{\hbar}{M}\operatorname{Re}\{i\oint_\Gamma f(z)\,dz\}\\&=\frac{\hbar}{M}\operatorname{Re}\{2\pi i\cdot i\sum_{\text{poles}\in\Gamma}\operatorname{Res}[f]\}.\end{aligned} \tag{A 5}$$

The complex function $f(z)$ has a simple pole with a residue of the value m at each of the points $z_2, z_3, \ldots, z_N$. Thus, according to (A 5), the line integral of the velocity field can be obtained by counting the number, N_r, of electrons within the circle. We then get the following expression for the paramagnetic contribution to the line integral

$$\oint\mathbf{v}_{\text{para}}\cdot d\mathbf{s}=\frac{\hbar}{M}\{-2\pi m N_r\} \tag{A 6}$$

which together with the diamagnetic contribution yields the circulation expressed in (6).

The circulation of the velocity field of the vortex state is obtained in a similar way:

$$\begin{aligned}\nabla_1\,\Psi_h&=\nabla_1\left[\prod_{i=1}^{N}z_i\,\Psi_{\nu=\frac{1}{m}}\right]=\left[\frac{\nabla_1 z_1}{z_1}+\frac{\nabla_1\psi_{\nu=\frac{1}{m}}}{\Psi_{\nu=\frac{1}{m}}}\right]\Psi_h\\&=\left[\left\{\frac{1}{z_1}+\sum_{k=2}^{N}\frac{m}{z_1-z_k}\right\}(+1,-i)\right.\\&\quad\left.+\left(\frac{-x_1}{2l^2},\frac{-y_1}{2l^2}\right)\right]\Psi_h.\end{aligned} \tag{A 7}$$

The velocity field of the vortex state is therefore obtained from the velocity field of the $\nu=\frac{1}{m}$-state by adding $\frac{1}{z}$ (having a simple pole at $z=0$ with the residue $+1$) to the function $f(z)$ introduced in (A 3). Thus we obtain

$$\begin{aligned}\oint\mathbf{v}_{\text{para}}\cdot d\mathbf{s}&=\frac{\hbar}{M}\operatorname{Re}\left\{2\pi i\cdot i\sum_{\text{poles}\in\Gamma}\operatorname{Res}\left[\frac{1}{z}+f(z)\right]\right\}\\&=\frac{\hbar}{M}\{-2\pi(1+mN_r^+)\}.\end{aligned} \tag{A 8}$$

By adding the diamagnetic contribution

$$\frac{e}{M}\Phi_r^+=\frac{\hbar}{M}\frac{\Phi_r^+}{\Phi_0}$$

and the paramagnetic contribution, (A 8), we obtain the circulation of the velocity field (10).

Appendix B

In this appendix we first calculate the density profile and the velocity field of the vortex state (9) at $\nu=1$. Then we use these results to estimate the magnetic flux generated by the vortex current.

At $\nu=1$, i.e. for $m=1$, the Laughlin ground state wavefunction, $\Psi_{\nu=\frac{1}{m}}$, can be shown to be identical to the Slater determinant

$$\Psi_{\nu=1}=\frac{1}{\sqrt{N!}}\begin{vmatrix}\phi_0(z_1)&\phi_0(z_2)&\ldots&\phi_0(z_N)\\\phi_1(z_1)&&\ldots&\phi_1(z_N)\\\vdots&&&\vdots\\\phi_{N-1}(z_1)&&\ldots&\phi_{N-1}(z_N)\end{vmatrix} \tag{B 2}$$

where

$$\phi_p(z)=\left(\frac{1}{2\pi 2^p p!\,l^2}\right)^{\frac{1}{2}}\left(\frac{z}{l}\right)^p\exp\left(-\frac{|z|^2}{4l^2}\right) \tag{B 2}$$

are the single particle states in the cylindrical gauge $\mathbf{A}=B\left(-\frac{y}{2},+\frac{x}{2},0\right)$.

The vortex wavefunction then takes the form

$$\Psi_h=K_h\prod_{j=1}^{N}z_i\,\Psi_{\nu=1} \tag{B 3}$$

where the normalization constant K_h can be found from

$$K_h^{-2} = \int \dots \int \mathrm{d}\mathbf{r}_1 \dots \mathrm{d}\mathbf{r}_N \prod_{i,k=1}^{N} z_i z_k^* \Psi_{\nu=1} \Psi_{\nu=1}^*$$

$$= \int \dots \int \mathrm{d}\mathbf{r}_1 \dots \mathrm{d}\mathbf{r}_N \prod_{i=1}^{N} |z_i|^2 |\Psi_{\nu=1}|^2 \tag{B4}$$

where $z = x - iy = r\mathrm{e}^{-i\theta}$. By use of (B1) we obtain

$$K_h^{-2} = \int \dots \int \mathrm{d}\mathbf{r}_1 \dots \mathrm{d}\mathbf{r}_N \prod_{i=1}^{N} r_i^2 \frac{1}{N!} \cdot \sum_{\sigma \in P} [|\phi_0(z_{\sigma(1)})|^2 |\phi_1(z_{\sigma(2)})|^2 \dots |\phi_{N-1}(z_{\sigma(N)})|^2] \tag{B5}$$

where the sum includes all permutations, σ, of the particles. Since each of the $N!$ integrals yields the same contribution we get

$$K_h^{-2} = \int \dots \int \mathrm{d}\mathbf{r}_1 \dots \mathrm{d}\mathbf{r}_N \prod_{i=1}^{N} r_i^2 \cdot [|\phi_0(z_1)|^2 |\phi_1(z_2))|^2 \dots |\phi_{N-1}(z_N)|^2]. \tag{B6}$$

Using that

$$\int \mathrm{d}\mathbf{r}\, r^2 |\phi_p(z)|^2 = \int_0^\infty 2\pi r\, \mathrm{d}r\, r^2 \frac{1}{2\pi 2^p p!\, l^2} \left(\frac{r^2}{l^2}\right)^p \exp\left(-\frac{r^2}{2l^2}\right) = \frac{2l^2}{p!} \int_0^\infty R^{p+1} \exp(-R)\, \mathrm{d}R = 2l^2(p+1) \tag{B7}$$

we find

$$K_h = \left(\frac{1}{(2l^2)^N N!}\right)^{\frac{1}{2}}. \tag{B8}$$

Then let us calculate the particle density $n(\mathbf{r})$ in the vortex region. Taking $\mathbf{r}_1 = \mathbf{r}$ we have

$$n(\mathbf{r}) = N \int \dots \int 2\pi r_2\, \mathrm{d}r_2 \dots 2\pi r_N\, \mathrm{d}r_N\, K_h^2 \prod_{i=2}^{N} \Psi_{\nu=1} \Psi_{\nu=1}^*$$

$$= N K_h^2 (N-1)!\, (2l^2)^{N-1} \frac{1}{\pi} \sum_{p=0}^{N-1} \frac{1}{(p+1)!} \cdot \left(\frac{r^2}{2l^2}\right)^{p+1} \exp\left\{-\frac{r^2}{2l^2}\right\} \tag{B9}$$

which by use of (B8) can be written as

$$n(\mathbf{r}) = \frac{1}{a_0}\left[-1 + \sum_{n=0}^{N} \frac{1}{n!}\left(\frac{r^2}{2l^2}\right)^n\right] \exp\left\{-\frac{r^2}{2l^2}\right\} \tag{B10}$$

where $a_0 = 2\pi l^2 = \frac{h}{eB}$. As long as we are well away from the edge of the sample (B10) takes the form

$$n(\mathbf{r}) = \frac{1}{a_0}\left[1 - \exp\left\{-\frac{r^2}{2l^2}\right\}\right]. \tag{B11}$$

Suppose, starting from the filled Landau level, that we remove an electron from the single particle state $\phi_0(z) = \left(\frac{1}{a_0}\right)^{\frac{1}{2}} \exp\{-|z|^2/4l^2\}$. We then obtain a state with the density given in (B11), but although Ψ_h has the same density as this state; it differs from it by possessing a velocity field, which can be calculated from the circulation of the velocity field (10). Using (B11) we calculate $\langle N_r^+ \rangle = \int_0^r n(\rho)\, 2\pi\rho\, \mathrm{d}\rho$ and insert the result into (10), whereby we obtain

$$v(r) = \frac{1}{2\pi r} \frac{h}{M}\left[\frac{\Phi_r^+}{\Phi_0} - \frac{r^2}{2l^2} - \exp\left\{-\frac{r^2}{2l^2}\right\}\right] = -\frac{1}{2\pi r} \frac{h}{M} \exp\left\{-\frac{r^2}{2l^2}\right\}. \tag{B12}$$

We can now calculate the magnetic field generated by the vortex current. In the following we use the cylindrical coordinate system $(\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z)$ with r and z measured in units of the magnetic length l. From (B11) and (B12) we find the following expression for the 3-dimensional vortex current density

$$\mathbf{J} = -\frac{e\, n(r)}{d_z} v(r)\, \mathbf{e}_\phi = \frac{eh}{(2\pi)^2 \mathrm{d}_z M l^3} \frac{1}{r} \cdot \left[\exp\left\{-\frac{1}{2}r^2\right\} - \exp\{-r^2\}\right] \mathbf{e}_\phi \tag{B13}$$

where d_z is the extension of the electronic wavefunctions in the z-direction. The cylindrical symmetry of $\mathbf{J}$ leads to an induced magnetic field of the form $\mathbf{b} = b(r)\, \mathbf{e}_z$. The rotation of $\mathbf{b}$ is then

$$\nabla \times \mathbf{b} = \begin{vmatrix} \frac{1}{lr}\mathbf{e}_r & \frac{1}{l}\partial_r & 0 \\ \mathbf{e}_\phi & \partial_\phi & 0 \\ \frac{1}{lr}\mathbf{e}_z & \frac{1}{l}\partial_z & b(r) \end{vmatrix} = -\frac{1}{l}\partial_r b(r)\, \mathbf{e}_\phi. \tag{B14}$$

From the Maxwell equation $\nabla \times \mathbf{b} = \mu_0 \mathbf{J}$, (B13), and (B14) we obtain

$$\partial_r b(r) = \beta \frac{1}{r}\left[\exp\{-r^2\} - \exp\left\{-\frac{1}{2} r^2\right\}\right] \tag{B15}$$

with

$$\beta = \frac{e h \mu_0}{(2\pi)^2 d_z M l^2}. \tag{B16}$$

The magnetic field, $b(R)$, can be found from (B15) using the boundary condition $\lim_{R \to \infty} b(R) = 0$:

$$\begin{aligned} b(R) &= \int_{\infty}^{R} \partial_r b(r)\, \mathrm{d}r \\ &= \beta \int_{R}^{\infty} \left(\frac{\exp\{-\frac{1}{2} r^2\}}{r} - \frac{\exp\{-r^2\}}{r}\right) \mathrm{d}r \\ &= \frac{\beta}{2}\left[E_1\left(\frac{1}{2} R^2\right) - E_1(R^2)\right] \end{aligned} \tag{B17}$$

where $E_1(x)$ is the exponential integral[2]. Taking $d_z \simeq 5\,nm$ and $M = 0.067 m_e$ it follows from (B16) that

$$\frac{\beta}{2} \simeq 8 \cdot 10^{-6}\, B_0 \tag{B18}$$

where B_0 is the mid-plateau value of the external magnetic field. In the centre of the vortex the magnetic field generated by the vortex current is $6 \cdot 10^{-6} B_0$, in the distance $3l$ from the centre it is $2 \cdot 10^{-8} B_0$, and at $5l$ it is only $2 \cdot 10^{-12} B_0$.

---

[2] $E_1(x) = \int_x^{\infty} \frac{e^{-u}}{u}\, \mathrm{d}u = -\gamma - \ln x + \sum_{n=1}^{\infty} \frac{(-x)^n}{n \cdot n!} \approx \frac{e^{-x}}{x} \sum_{n=0}^{\infty} \frac{n!}{(-x)^n}$

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H. Bruus
Danish Institute of Fundamental Metrology
Building 322
Lundtoftevej 100
DK-2800 Lyngby
Denmark

O.P. Hansen, E.B. Hansen
Physics Laboratory
University of Copenhagen
H.C. Ørsted Institute
Universitetsparken 5
DK-2100 Copenhagen Ø
Denmark