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LETTER TO THE EDITOR

The mechanism of plateau formation in the fractional quantum Hall effect

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Abstract. Laughlin's fractionally charged quasi-holes and quasi-electrons are assumed to be pinned, and to be subject to a force $j \times \Phi_0$ from the transport current. A force balance argument then explains the existence of Hall plateaus.

The explanation of plateau formation in the integer quantum Hall effect (IQHE) is based on the existence of a mobility gap between extended states in different Landau levels and single-particle localisation associated with this gap. For the fractional quantum Hall effect (FQHE) it has been conjectured that, analogously, localisation of quasi-particles might account for the finite step width (Girvin 1987, Laughlin 1987).

In the present Letter we propose a mechanism of plateau formation which supports the view that plateaus arise because of localisation of quasi-particles, i.e., Laughlin's fractionally charged quasi-holes and quasi-electrons.

To facilitate our argumentation, which is of a hydrodynamic nature, we shall use the notions vortices and anti-vortices instead of quasi-holes and quasi-electrons. The vortices and the anti-vortices, which have opposite directions of rotation, are created in numbers proportional to the deviation of the magnetic flux from its mid-plateau value: vortices for positive deviation, and anti-vortices for negative deviation.

The vortices (anti-vortices) are expected to remain pinned (Girvin 1987, Laughlin 1987) in the presence of a transport current that does not exceed the critical current where the FQHE breaks down. We assume that the vortices (anti-vortices) are subject to a force $j \times \Phi_0$, where j is the current density, $|\Phi_0| = h/e$, and where Φ_0 is parallel (antiparallel) to the magnetic field for vortices (anti-vortices). By inclusion of the reaction force from the pinning centres in a force balance equation for the electron system, we show that the Hall response, V_H, is independent of the magnetic field in an interval around $B_\nu = 1/m$.

Our picture is the following: suppose, for a two-dimensional system with N electrons at $T = 0$, we increase the magnetic field, B, relative to say $B_{\nu=\frac{1}{3}}$. The perfect matching of electrons and magnetic flux quanta (crucial to the correlations of the $\nu = \frac{1}{3}$ state) can then no longer be maintained everywhere. However, the electron liquid will try to maintain the favourable correlations of the $\nu = \frac{1}{3}$ state, and as a consequence will yield only locally to the increase of magnetic field. Thus defects in the form of vortices will arise around in the liquid (Haldane 1987, Laughlin 1987). Except for a change of magnetic length these vortices are equivalent to Laughlin's hole-like excitations at $\nu = \frac{1}{3}$. A detailed discussion of vortices is given below.

We first consider the case where the magnetic flux, Φ, has a mid-plateau value, i.e., $\Phi = mN\Phi_0$, with m being an odd integer. Keeping all electrons, except electron 1, at fixed positions the velocity field, $\boldsymbol{v}_1$, of electron 1 is

$$\boldsymbol{v}_1 = \boldsymbol{S}_1/\Psi^*\Psi \tag{1}$$

where $\boldsymbol{S}_1$ is the probability current density. Using Laughlin's ground-state wavefunction (Laughlin 1983) we find

$$\boldsymbol{v}_1 = \frac{m\hbar}{M}\sum_{j=2}^{N}\left[+\mathrm{Im}\left(\frac{1}{z_1 - z_j}\right); -\mathrm{Re}\left(\frac{1}{z_1 - z_j}\right)\right] + \frac{e}{M}\boldsymbol{A} \tag{2}$$

where M denotes the electron mass, N is the number of electrons, and $z_k = x_k - \mathrm{i}y_k$ are the complex electron coordinates (corresponding to an arrangement where the magnetic field points in the z direction). We calculate the line integral of $\boldsymbol{v}_1$ along a circle centred at the origin and with radius r:

$$\oint \boldsymbol{v}_1 \cdot \mathrm{d}\boldsymbol{s}_1 = \frac{h}{M}\left(-mN_r + \frac{\Phi_r}{\Phi_0}\right) \tag{3}$$

where N_r is the number of electrons with position inside the circle, and Φ_r is the flux enclosed by the circle. On the average the number of electrons within the circle is $\langle N_r \rangle = (1/m)\Phi_r/\Phi_0$ showing that

$$\left\langle \oint \boldsymbol{v}_1 \cdot \mathrm{d}\boldsymbol{s}_1 \right\rangle = 0. \tag{4}$$

If (4) was not fulfilled large 'hydrodynamical' energies would come into play (Halperin 1983). The mid-plateau state is different from neighbouring states by being without vorticity.

Now consider the situation where the total flux has been increased by one flux quantum as compared to the mid-plateau value $\Phi = mN\Phi_0$. In this case the ground state can qualitatively be described by (Haldane 1987)

$$\Psi_h = \prod_{j=1}^{N} z_j \prod_{i>k}^{N} (z_i - z_k)^m \exp\left(-\sum_{n=1}^{N} \frac{|z_n|^2}{4l^2}\right) \tag{5}$$

which, except for a minute change in magnetic length, is identical to Laughlin's quasi-hole excitation at $\nu = 1/m$. Calculating the circulation of $\boldsymbol{v}_1$ for this state we obtain

$$\oint \boldsymbol{v}_1 \cdot \mathrm{d}\boldsymbol{s}_1 = \frac{h}{M}\left(-1 - mN_r^+ + \frac{\Phi_r^+}{\Phi_0}\right) \tag{6}$$

where N_r^+ is the number of electrons inside the circle, and where Φ_r^+ is the flux enclosed by the circle. Taking the radius, r, to be a few magnetic lengths, we have that Φ_r^+ is essentially equal to Φ_r, while, according to Laughlin, $\langle N_r^+ \rangle = \langle N_r \rangle - (1/m)$, showing that $\langle \oint \boldsymbol{v}_1 \cdot \mathrm{d}\boldsymbol{s}_1 \rangle = 0$. In the limit $r \ll l$ we obtain $\langle \oint \boldsymbol{v}_1 \cdot \mathrm{d}\boldsymbol{s}_1 \rangle = -h/M$ displaying the clockwise rotation of the liquid in the vortex region.

Finally we consider what happens to the electron liquid when the flux is reduced by one flux quantum. Qualitatively the situation can be described by Laughlin's quasi-electron wavefunction with the appropriate magnetic length

$$\Psi_p = \left(\prod_{j=1}^{N} \frac{\partial}{\partial z_j} \prod_{i>k}^{N} (z_i - z_k)^m\right) \exp\left(-\sum_{n=1}^{N} \frac{|z_n|^2}{4l^2}\right). \tag{7}$$

This wavefunction also describes a rotating structure, an anti-vortex, but now with a local surplus of $1/m$ of an electron charge in the defect region. It follows from (7) that the anti-vortex rotates counter clockwise.

The densities of a vortex and an anti-vortex at $\nu = \frac{1}{3}$ are displayed in Laughlin (1987).

Qualitative features of a vortex can be elucidated by studying (5) for $m = 1$, because in this case it is possible to obtain analytical expressions for the velocity field and the electron density in the vortex region

$$v(r, \theta) = -(h/M)(1/2\pi r) \exp(-\pi r^2/a_0) e_\theta$$

$$n(r, \theta) = (1/a_0)[1 - \exp(-\pi r^2/a_0)]$$

where $a_0 = 2\pi l^2 = h/eB$.

The details of our picture are thus the following: as the flux is increased (decreased) relative to the mid-plateau value, $mN\Phi_0$, by a given numberof flux quanta the electron liquid responds by forming the same number, $V_\mathrm{v}(V_\mathrm{a})$, of vortices (anti-vortices):

$$\begin{aligned} V_\mathrm{v} &= \Phi/\Phi_0 - mN \qquad & B > B_{\nu=1/m} \\ V_\mathrm{a} &= mN - \Phi/\Phi_0 \qquad & B < B_{\nu=1/m}. \end{aligned} \tag{8}$$

As more and more vortices (anti-vortices) are created in the liquid the order in terms of correlation of the $\nu = 1/m$ state gradually breaks down. Thus (8) holds only in a certain interval around the plateau centre. Recently promising progress has been made as regards the establishment of a Ginzburg–Landau (GL) theory of the FQHE (Girvin 1987, Girvin and MacDonald 1987). A more comprehensive description of the vortex state probably will require a GL theory.

We shall next discuss the force balance in the presence of a transport current. Consider a vortex bound to a pinning centre. If a current is passed through the sample the vortex will experience a sort of Kutta–Joukowski force or rather a Magnus force. We shall assume that this force has the form†

$$f = j \times \Phi_0 \tag{9}$$

where j is the current density.

Being pinned the vortex cannot move. Therefore, the principle of action and reaction implies that the pinning centre acts on the electron system with the force $-f$. Except for the change of sign due to the opposite direction of rotation the same argument applies to a pinned anti-vortex.

We use the geometry shown in figure 1: the magnetic field, B, points in the z direction. A transport current, I, is passed through the sample in the x direction, thus giving rise to a magnetic force (Lorentz force), F_M, and an electric force, F_E, on the electron system.

† Hydrodynamic (momentum transfer) considerations support that $f = \alpha j \times \Phi_0$ with $\alpha = 1$, but the proof that $\alpha = 1$ is still lacking. We expect that (9) will find its justification within the framework of a GL theory.

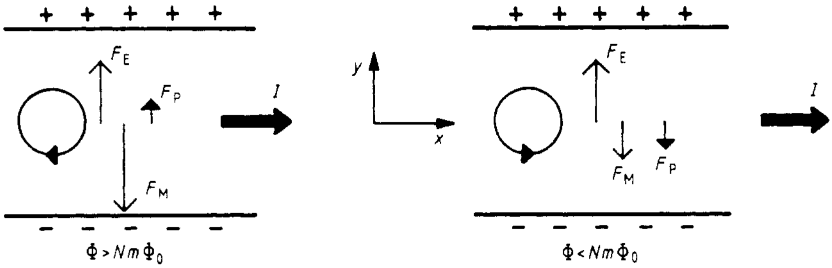

Figure 1. The force balance for the vortex state and the anti-vortex state. The clockwise and anticlockwise circles represent the vortices and anti-vortices. The electric force, F_E, is the same in the two cases. In vortex state ($\Phi > Nm\Phi_0$) the electric force, F_E, and the upward force, F_P, from the pinning centres, balance the magnetic force F_M. In the anti-vortex state ($\Phi < Nm\Phi_0$) F_P points downwards and together with F_M balances F_E.

The width of the sample in the y direction is denoted w. From (8) and (9) it follows that the pinning centres exert a force

$$F_P = (\Phi/\Phi_0 - mN)\Phi_0(I/w) \tag{10}$$

on the electron system. The force F_P is positive (pointing in the y direction) for $B > B_{\nu=1/m}$ and negative for $B < B_{\nu=1/m}$.

Since, in the QHE experimental set-up, the transverse current is zero, the total transverse force on the electron system must be zero:

$$0 = F_M + F_E + F_P \tag{11}$$

with $F_M = -\Phi(I//w)$ and $F_E = Ne(V_H/w)$, where V_H is the Hall voltage. Inserting these expressions and (10) into (11) we obtain

$$\Phi(I/w) = Ne(V_H/w) + (\Phi/\Phi_0 - mN)\Phi_0(I/w) \tag{12}$$

which implies

$$V_H = m(\Phi_0/e)I = m(h/e^2)I. \tag{13}$$

Thus, V_H comes out as independent of the magnetic flux, i.e. as a Hall plateau.

In order to generalise the preceding discussion we assume that at any canonical fraction ν_0 (that is a fraction around which a plateau is observed) the electron system is in a state which resembles the $\nu = \frac{1}{3}$ state in the sense that the electron liquid responds to increase (decrease) of magnetic field by forming vortices (anti-vortices) according to the following generalisation of (8):

$$\begin{aligned} V_v &= \Phi/\Phi_0 - (1/\nu_0)N \qquad && B > B_{\nu_0} \\ V_a &= (1/\nu_0)N - \Phi/\Phi_0 \qquad && B < B_{\nu_0}. \end{aligned} \tag{14}$$

These expressions lead to a pinning force which by substitution into (12) produces plateaus. The possibility of extending the theory to the IQHE is being investigated.

Given the validity of the vortex picture and the correctness of (9) we have shown that the formation of plateaus in FQHE is due to localisation of quasi-particles, meaning pinning of vortices and anti-vortices.

References

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