J. Phys. A: Math. Gen. **29** (1996) L483–L488. Printed in the UK

LETTER TO THE EDITOR

Spectral properties of statistical mechanics models

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Received 9 April 1996

Abstract. The full spectrum of transfer matrices of the general eight-vertex model on a square lattice is obtained by numerical diagonalization. The eigenvalue spacing distribution and the spectral rigidity are analysed. In non-integrable regimes we have found eigenvalue repulsion as for the Gaussian orthogonal ensemble in random matrix theory. By contrast, in integrable regimes we have found eigenvalue independence leading to a Poissonian behaviour, and, for some points, level clustering. These first examples from classical statistical mechanics suggest that the conjecture of integrability successfully applied to quantum spin systems also holds for classical systems.

Since the work of Wigner [1], random matrix theory (RMT) has been applied successfully in various domains of physics [2]. Recently, several quantum spin Hamiltonians have been investigated from this point of view. It has been found [3, 4] that one-dimensional systems for which the Bethe ansatz applies have a level spacing distribution close to a Poissonian (exponential) distribution, $P(s) = \exp(-s)$, whereas if the Bethe ansatz does not apply, the level spacing distribution is described by the Wigner surmise for the Gaussian orthogonal ensemble (GOE), $P(s) = (\pi/2)s\exp(-\pi s^2/4)$. Similar results have been found for two-dimensional systems [5]. This suggests that the GOE describes properly *some* properties of the spectrum of complex quantum systems. In this letter we extend the RMT analysis from quantum spin systems to models of classical statistical mechanics. In particular, we look at possible consequences of integrability on the spectral properties of the transfer matrices.

At first sight it seems natural to start with the Ising model in two dimensions without magnetic field as an example of an integrable model, and then to add a magnetic field. It turns out that the spectrum of transfer matrices of the Ising model for finite size leads to numerical difficulties as explained below. We then have chosen the case of the general eight-vertex model on a square lattice (which contains the zero-field Ising model as a special case) [6]. Moreover, it is known that zero-field eight-vertex transfer matrices commute with the Hamiltonian of the anisotropic XYZ quantum spin chain for certain relations between the parameters of the two models [7]. We shall use the notation of [6] to designate the eight admissible vertices and their respective Boltzmann weights a, a', b, b', c, c', d, and d'. We consider the row-to-row transfer matrices T_N to build iteratively a periodic rectangular lattice by adding rows of length N with periodic boundary conditions. Therefore, the partition function of a periodic rectangle of n rows of N sites is $Z_{n,N} = \mathrm{Tr}(T_N^n)$. Note that there can be different expressions for the matrix T_N, but all these expressions have the same value of the trace of the nth power for any n, and, therefore, they can be deduced from each other by a similarity transformation; the spectrum is indeed an intrinsic quantity which does not depend on any particular choice of the transfer matrix. To perform the usual statistical

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0305-4470/96/180483+06$19.50

analysis of the spectrum we need to have real eigenvalues. However, in general, this is not the case and we will restrict ourselves to cases where the transfer matrix is symmetric. It is well known that the eight-vertex model can be mapped onto an anisotropic Ising model on a square lattice with diagonal interactions and four spin interactions around each plaquette. We again use the notations of [6] and introduce the five coupling constants J_h, J_v, J, J', and J''. The transfer matrix of the spin model can be chosen symmetric if $J = J'$. In terms of the Boltzmann weights of the transfer matrix it requires only that $c = d = c' = d'$ ($c = c'$ and $d = d'$ is not a restriction). This condition is verified for models without electrical field (i.e. when $a = a'$ and $b = b'$) and also for models with a field. So we are able to build symmetric transfer matrices for integrable cases with $a = a'$, $b = b'$, and $c = c' = d = d'$, as well as for non-integrable cases with $a \neq a'$, $b \neq b'$, and $c = c' = d = d'$.

Before presenting our results, we briefly recall some features of the RMT analysis, which is a statistical analysis of the eigenvalues of a given matrix regarded as an ordered set. First, one has to sort the eigenvalues according to the symmetry of the corresponding eigenstate. In contrast to quantum spin systems, transfer matrices posses *a priori* no continuous symmetry (as e.g. the SU(2) spin symmetry), but only space symmetries. For row-to-row transfer matrices, these are given by the automorphy group of a single row (and not of the full lattice). This group is the set of permutations g of sites such that $g(i)$ and $g(j)$ are neighbours if and only if i and j are neighbours. To each g acting on the set of vertices one can easily associate a linear operator $\hat{g}$ acting in the configuration space. Obviously $\hat{g}$ and T_N commute. It is then possible to construct a set of projectors onto invariant subspaces of T_N. This amounts to block-diagonalizing T_N. This is not only a useful way of lowering the size of the matrices which have to be diagonalized, but also a useful manner to sort the eigenvalues. The automorphy group of the periodic ring of length N is the dihedral group D_N generated by a translation and a reflection. Elementary group theoretical analysis can be performed to build the $((N/2) + 3$ if N is even, or $(N - 1)/2 + 2$ if N is odd) projectors onto the invariant subspaces. The transfer matrix of the zero-field model is also invariant under the reversal of all arrows of the vertices and another projector has to be applied. Second, to find universal behaviour within each invariant subspace, one needs to 'rescale' the eigenvalues E_i in order to have a *local* density of eigenvalues equal to one. This operation is called the 'unfolding' and produces the unfolded eigenvalues ϵ_i. The aim is to remove the non-universal or system-specific large-scale variations of the integrated density of states, and to study only the presumably universal short-scale fluctuations. It amounts to computing an average integrated density of states $N_{\mathrm{av}}(E)$ which is the smooth part of the actual integrated density of states. We then have $\epsilon_i = N_{\mathrm{av}}(E_i)$. In the generic case, several methods can be used to compute $N_{\mathrm{av}}(E)$: running average unfolding (local averaging of eigenvalues followed by interpolation), Gaussian unfolding (Gaussian broadening of each delta peak in the density of states), and Fourier unfolding (removal of short-scale wavelengths using Fourier transformation).

The simplest quantity one calculates in RMT analysis is the distribution $P(s)$ of the differences between two consecutive unfolded eigenvalues $s_i = \epsilon_{i+1} - \epsilon_i$. For integrable systems a Poissonian distribution is expected, since the ϵ_i are expected to be independent. In contrast, for the simplest non-integrable systems the Wigner surmise is expected. Another quantity of interest is the spectral rigidity [2]:

$$\Delta_3(L) = \left\langle \frac{1}{L} \min_{a,b} \int_{\alpha - L/2}^{\alpha + L/2} (N(\epsilon) - a\epsilon - b)^2 \, \mathrm{d}\epsilon \right\rangle_\alpha$$

where $\langle \ldots \rangle_\alpha$ denotes an average over α. This quantity measures the deviation from equal spacing. For a totally rigid spectrum, as that of the harmonic oscillator, one has

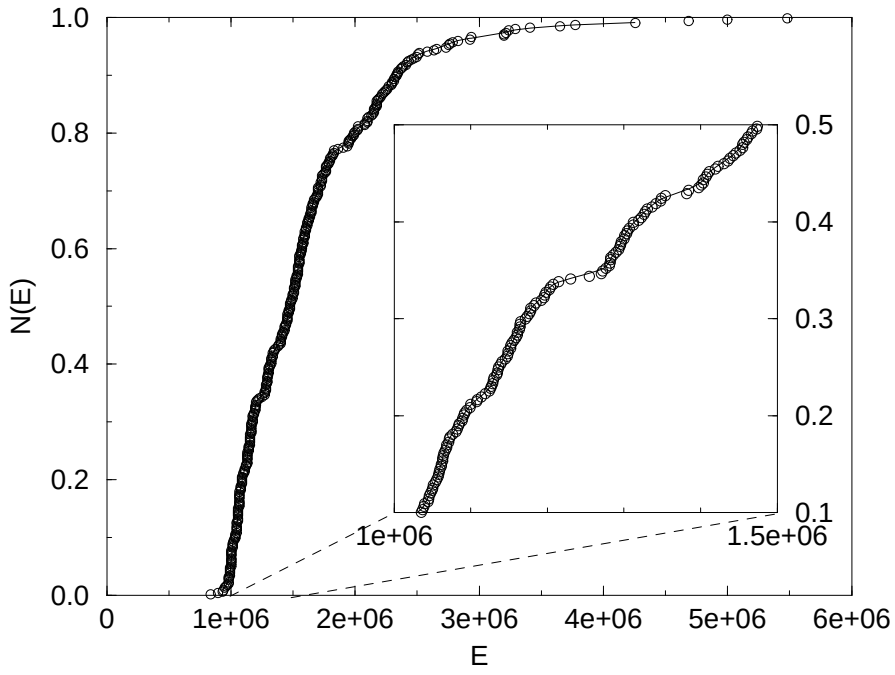

Figure 1. The integrated density of eigenvalues $N(E)$ (circles) for the case $a = 2.5$, $a' = 1.6$, $b = b' = 3$, $c = c' = d = d' = 1/\sqrt{6}$ for a single symmetry invariant block. The insert shows how complex $N(E)$ is, even at a very fine scale. The full curve is $N_{\mathrm{av}}(E)$.

$\Delta_3^{\mathrm{osc}}(L) = \frac{1}{12}$, for an integrable (Poissonian) system one has $\Delta_3^{\mathrm{Poi}}(L) = L/15$, while for the Gaussian orthogonal ensemble one has $\Delta_3^{\mathrm{GOE}}(L) = (1/\pi^2)(\log(L) - 0.0687) + \mathcal{O}(L^{-1})$. It has been found that the spectral rigidity of quantum spin systems follows $\Delta_3^{\mathrm{Poi}}(L)$ in the integrable case and $\Delta_3^{\mathrm{GOE}}(L)$ in the non-integrable case. However, in both cases, even though $P(s)$ is in good agreement with RMT, deviations from RMT occur for $\Delta_3(L)$ at some system-dependent point L^*. This stems from the fact that the rigidity $\Delta_3(L)$ probes correlations beyond nearest neighbours in contrast to $P(s)$. This is probably why the rigidity is much more sensitive to the parameters of the unfolding than the spacing distribution.

We have generated transfer matrices for different values of the Boltzmann weights and linear size $N = 16$ (respectively $N = 14$). This leads to matrix sizes of $65\,536^2$ (respectively, $16\,384^2$). Because of the eight-vertex condition, the transfer matrix couples only configurations with the numbers of up (or down) arrows having the same parity, so the matrix trivially separates into two blocks. After projection the matrix splits up into 18 (respectively, 16) symmetry invariant blocks of which the largest has a size of 2062^2 (respectively, 594^2). We have tried several methods of unfolding. The results presented here are obtained using either a Gaussian unfolding with a local broadening over five states, or a running average unfolding over ten states. In figure 1 we present a typical integrated density of eigenvalues together with the averaged curve. The spectrum is seen to be rather more complex than a usual spectrum of a quantum spin system, and thus the unfolding has to be performed very carefully. Figure 2 shows the probability distribution

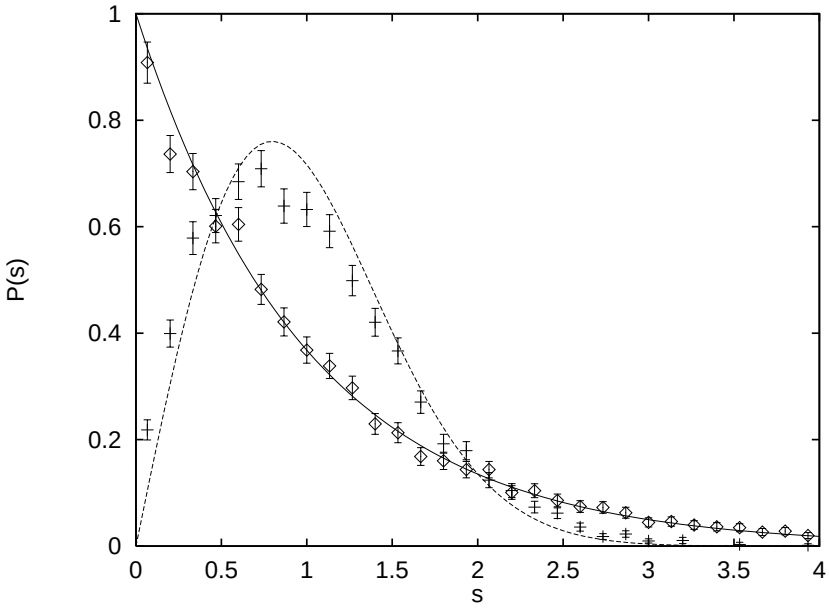

Figure 2. The eigenvalue spacing distribution $P(s)$ for two row-to-row transfer matrices of the eight-vertex model with $N = 14$. The diamonds correspond to $a = a' = 2$, $b = b' = 3$, $c = c' = d = d' = 1/\sqrt{6}$: this corresponds to an integrable point in the ordered region of the phase diagram without electrical field. The plus signs correspond to $a = 2.5$, $a' = 1.6$, $b = b' = 3$, $c = c' = d = d' = 1/\sqrt{6}$. The full curve is the Poissonian distribution while the broken curve is the Wigner surmise.

of the eigenvalue spacings averaged over all representations for two representative sets of Boltzmann weights. The diamonds correspond to a zero-field case with $a = a' = 2$, $b = b' = 3$, and $c = c' = d = d' = 1/\sqrt{6}$. This point lies in an ordered region of the phase diagram, and the transfer matrix has eigenvectors of the Bethe ansatz form. The spacing distribution is close to the Poissonian distribution. The plus signs on the same figure correspond to the case $a = 2.5$, $a' = 1.6$, $b = b' = 3$, and $c = c' = d = d' = 1/\sqrt{6}$. This point is in a region of parameter space where the Bethe ansatz does not apply. The spacing distribution is close to the Wigner surmise. In particular, the level repulsion is clearly seen.

To test further the idea that integrability leads to a Poisson law and non-integrability leads to the Wigner surmise we have also calculated the spectrum of transfer matrices for some particular points satisfying the free-fermion condition $aa' + bb' = cc' + dd'$ [8]. In this case the results are less clear. In figure 3 we have chosen a free-fermion point within the generally non-integrable region where an electrical field is present. The Boltzmann weights are $a = 0.8$, $a' = 1/a$, $b = b' = \sqrt{2c^2 - 1}$, and $c = c' = d = d' = 1/1.1$. The spacing distribution is close neither to the Poissonian distribution nor to the Wigner surmise. In particular, there is a peak at $s = 0$ indicating level clustering. We have found the same phenomenon for the pure two-dimensional Ising model which can be mapped onto an eight-vertex model satisfying the free-fermion condition. This behaviour is usually found together with a very involved density of eigenvalues and leads to numerical difficulties in the unfolding. A possible explanation of the peak at $s = 0$ could be that the free-fermion model is a trivial non-generic model as, for example, the Hubbard model at zero Coulomb

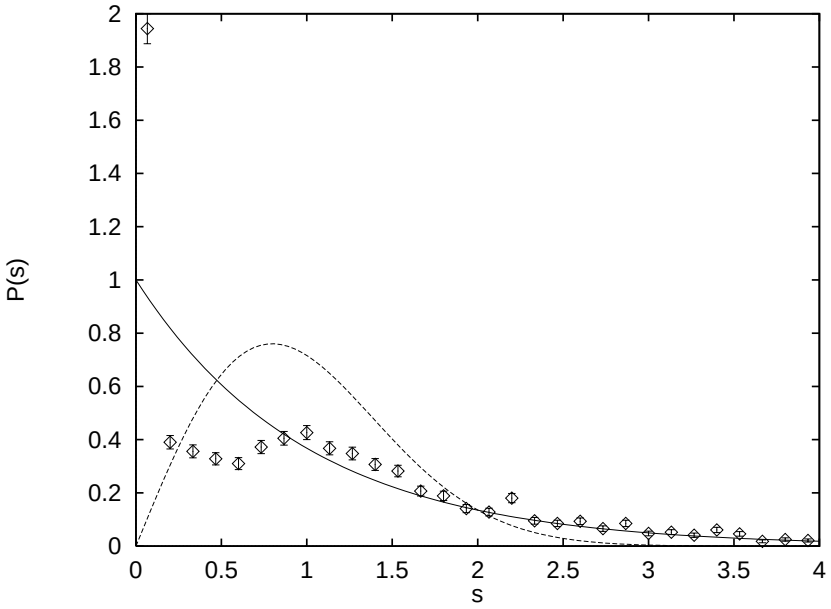

Figure 3. The eigenvalue spacing distribution $P(s)$ for a point satisfying the free-fermion condition. The Boltzmann weights are $a = 0.8$, $a' = 1/a$, $b = b' = \sqrt{2c^2 - 1}$, and $c = c' = d = d' = 1/1.1$. Note the peak near $s = 0$ and the Poissonian tail for $s > 1$.

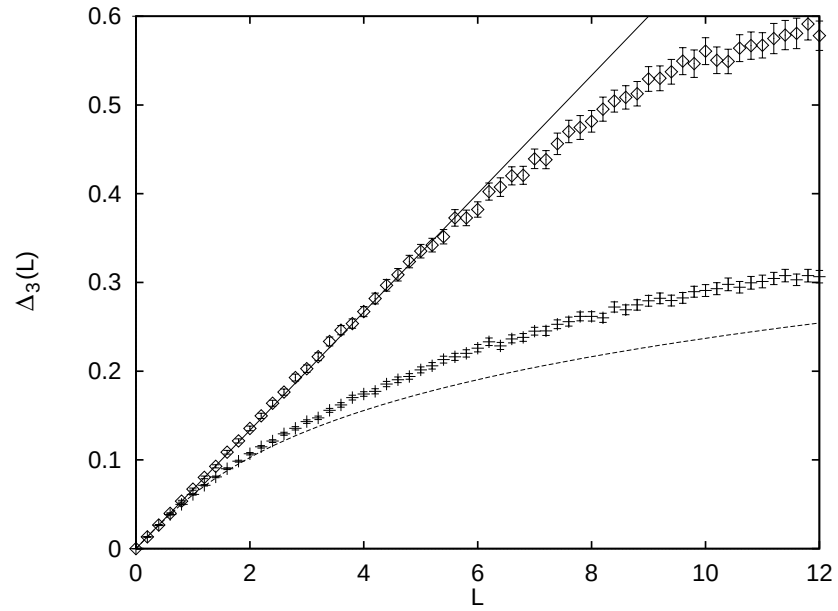

Figure 4. The spectral rigidity $\Delta_3(L)$ for the same cases as in figure 2.

repulsion. However, for some other values of the Boltzmann weights also obeying the free-fermion condition, the spacing distribution is much closer to the Poissonian distribution. This suggests as another possible explanation the existence of quasi-degeneracy leading to

a Shnirelman peak at the origin [9] for this specific set of Boltzmann weights. This will be studied in detail in a forthcoming publication. From the above results we conjecture that the spacing distribution of eigenvalues of non-integrable models is close to the Wigner surmise corresponding to level repulsion, while for integrable models there is no level repulsion. In integrable systems there is level independence leading to a Poissonian spacing distribution, but with a tendency to level attraction in some cases.

To go further in the RMT analysis, we present in figure 4 the spectral rigidity $\Delta_3(L)$ for the same points in parameter space corresponding to integrability and to non-integrability as in figure 2. The two limiting cases corresponding to the Poissonian distributed eigenvalues (full curve) and to GOE distributed eigenvalues (broken curve) are also shown. For the integrable point the agreement between the numerical data and the expected rigidity is good up to a value $L \approx 5$. For larger values of L a saturation occurs showing the limitation of the model of independent eigenvalues. For the non-integrable case the departure of the rigidity from the expected behaviour appears at $L \approx 2$, indicating that the RMT is only valid at short scales. Such behaviour has already been seen in quantum spin systems [5]. We stress that these numerical results depend much more on the unfolding than the results concerning the spacing distribution.

In summary, we have numerically calculated the spectrum of transfer matrices of the two-dimensional eight-vertex model for various parameters. After having sorted and unfolded the spectrum we have computed the eigenvalue spacing distribution and the spectral rigidity averaged over all representations. To our knowledge this is the first RMT analysis of transfer matrices of models in classical statistical mechanics. We have found that the non-integrable cases are well described at short scales by the Gaussian orthogonal ensemble, while in the integrable cases the eigenvalues are mostly independent. We speculate that this is a general result.

HM and J-CAd'A would like to thank JM Maillard for many discussions concerning integrability of vertex models. HB is supported by the European Commission under grant No ERBFMBICT 950414.

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