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Magnetoconductivity in disordered quantum wires

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Abstract. We generalize the linear response theory of magnetoconductivity of disordered two-dimensional electron gases to include the case where a confinement potential is present in one of the two spatial directions. The theory allows studies of the effect of suppression of backscattering due to the existence of spatially separated edge states in high magnetic fields. Explicit calculations of the conductivity are carried out for weak scattering in two limits: the weak and the strong confinement limit.

1. Introduction

Since the discovery of the importance of edge states for transport properties of quasi-two- and one-dimensional electron gases in high magnetic fields [1–5] it has been of importance to construct a microscopic theory taking the edge states explicitly into account, as anticipated by Halperin [6]. Most of the recent theoretical work devoted to the study of magnetotransport has been based on the Landauer–Büttiker formalism [5, 7–10]. The general attitude in these papers has been to make phenomenological predictions in the sense that instead of calculating the t-matrix, as required by the theory, various *ad hoc* assumptions have been made. Our approach differs from this general trend: in this study of the magnetoconductivity of disordered quantum wires we employ the Kubo formalism and, in close analogy with the work of Ando and Uemura [11, 12] which is free from phenomenological assumptions, we calculate the magnetoconductivity directly from a model Hamiltonian of our system. The main result of this paper is the extension of the Ando–Uemura calculation of the bulk magnetoconductivity of a disordered two-dimensional electron gas (2DEG) to include the case where a confinement potential is present in one of the two spatial directions, allowing a study on the microscopic level of the effect of the suppression of backscattering due to the existence of edge states in high magnetic fields and the spatial separation between states with opposite velocities. To obtain the magnetoconductivity in a consistent way we evaluate the vertex corrections of the conductivity bubble diagram. It turns out that the essential physics—the suppression of backscattering—is contained in the vertex corrections. Such a calculation is equivalent to a detailed calculation of $\mathrm{Tr}(\mathbf{t}^\dagger\mathbf{t})$ in the Landauer–Büttiker formalism if one would like to go beyond the above-mentioned phenomenological considerations. We derive the general equations for the magnetoconductivity which are then solved

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in the limit of a low scattering strength. In this limit our results are equivalent to those found by using the Boltzmann transport equation for scattering [13, 14].

The paper is organized as follows. In section 2 the model Hamiltonian for an impurity-free quantum wire is introduced, and the unperturbed eigenstates are discussed. Section 3 outlines the standard impurity scattering formalism in the self-consistent Born approximation (SCBA). In section 4 the general expression for the magnetoconductivity is derived and explicit results are calculated in the weak and in the strong confinement potential limits. Summary, discussion and conclusion are offered in section 5. Mathematical details are left for two appendices: appendix A describes the general conditions that must be satisfied by the vertex functions (the Ward identities), and appendix B derives the formula for the conductivity used in section 4.

2. The model of the impurity free quantum wire

We consider a model for a spinless 2DEG with an arbitrarily shaped confinement potential in one direction. Only the case of a parabolic confinement potential can be solved analytically, but we give analytical expressions for approximate eigenstates in the high-field limit. The parabola is relevant for narrow (submicron) channels whereas due to the screening the potential for a wider sample is flat in the middle. For a harmonic confinement potential all states are affected by the potential, and one cannot talk about bulk and edge states as two distinct types of states. However, the parabolic potential gives results which have the same qualitative features as a more realistic confinement potential for wider channels.

Imagine a wire along the x-direction confined in the y-direction. Let the length of the wire be denoted L. The magnetic field points along the z-direction. Our spinless single-electron Hamiltonian for the impurity-free quantum wire is

$$\mathcal{H} = (1/2m)\,(\boldsymbol{p} + e\boldsymbol{A})^2 + V(y). \tag{1}$$

We use the Landau gauge: $\boldsymbol{A} = -B(y,0,0)$, which, combined with translational invariance in the x-direction, yields eigenstate wavefunctions of the form

$$\psi(x,y) = (1/\sqrt{L})\mathrm{e}^{\mathrm{i}kx}\langle y|Nk\rangle. \tag{2}$$

Here $\langle y|Nk\rangle$ is the solution to the effective Schrödinger equation in the y-coordinate, $\mathcal{H}_k\langle y|Nk\rangle = E_{Nk}\langle y|Nk\rangle$. The Hamiltonian H_k is given by

$$\mathcal{H}_k = \tfrac{1}{2}\hbar\omega_c(-\ell_c^2(\mathrm{d}^2/\mathrm{d}y^2) + (y/\ell_c - k\ell_c)^2) + V(y) \tag{3}$$

where $\omega_c = eB/m$ and $\ell_c^2 = \hbar/eB$. Before proceeding with the approximate solution we show the general result for the matrix element of the current operator, which enters the conductivity formula used later on. Omitting all details, the result follows from the formulae derived in appendix A:

$$\langle Nk|j_x|N'k'\rangle = ev_{Nk}\delta_{kk'}\delta_{NN'} + (e/\hbar)(E_{Nk} - E_{N'k})\langle Nk|(\partial/\partial k)|N'k\rangle\delta_{kk'} \tag{4}$$

$$\langle Nk|j_y|N'k'\rangle = \mathrm{i}(e/\hbar)(E_{Nk} - E_{N'k})\langle Nk|y|N'k\rangle\delta_{kk'} \tag{5}$$

where

$$v_{Nk} = (1/\hbar)\partial E_{Nk}/\partial k \tag{6}$$

is the drift velocity of the eigenstates $|Nk\rangle$. The interpretation is clear: the first term in j_x corresponds to the current carried by the edge states, which have a non-vanishing drift velocity, whereas the second term in j_x (and also j_y) is more 'bulk-like' in the sense that charge is moved by scattering between different Landau levels.

In this paper we will consider the case of a strong magnetic field. Consequently the spatial extent of the wave functions is small for the lower Landau levels, and hence it is a good approximation to expand the potential around the centre of the wave function. This centre coordinate is located at the k-dependent position, y_k, where the derivative of the total potential in the Hamiltonian vanishes:

$$\hbar\omega_c(y_k/\ell_c^2 - k) + V'(y_k) = 0. \tag{7}$$

We note that for a constant potential $y_k = k\ell_c^2$. Second-order Taylor expansion in y around y_k yields the following form of $\mathcal{H}_k$:

$$\mathcal{H}_k \approx \tfrac{1}{2}\hbar\omega_c\left(-\ell_c^2(\partial^2/\partial y^2) + (y - y_k)^2(1 + \gamma_k)/\ell_c^2\right) + V(y_k) + \tfrac{1}{2}mv_k^2. \tag{8}$$

Here we have introduced $\gamma_k \equiv V''(y_k)\ell_c^2/\hbar\omega_c$, and v_k is the drift velocity:

$$v_k = (y_k - k\ell_c^2)\omega_c = -V'(y_k)/eB. \tag{9}$$

This is the classical drift velocity of an electron moving along the channel at a distance y_k from the bottom of the potential trough, and it follows simply from balancing the Lorentz force and that of the confinement potential. The drift velocity is also the group velocity of the eigenstate centred at y_k as was shown above.

The approximate (though for a parabolic potential exact) eigenstates are the harmonic oscillator wave functions which we denote $|nk\rangle$ (later on we use the notation $|Nk\rangle$ for the eigenstates of the Hamiltonian with a general potential). The corresponding eigenenergies are

$$E_{nk} = \hbar\omega_k\left(n + \tfrac{1}{2}\right) + V(y_k) + \tfrac{1}{2}mv_k^2 \qquad n = 0, 1, 2, \ldots \tag{10}$$

where $\omega_k = \omega_c\sqrt{1 + \gamma_k}$ is the renormalized oscillator frequency. The eigenstates are given by

$$\langle \mathbf{r}|nk\rangle = \frac{1}{\sqrt{L}}e^{ikx}\frac{1}{\sqrt{\sqrt{\pi}2^n n!\ell_k}}H_n\left(\frac{y - y_k}{\ell_k}\right)\exp\left(-\frac{(y - y_k)^2}{2\ell_k^2}\right) \tag{11}$$

where $\ell_k^2 = \ell_c^2/\sqrt{1 + \gamma_k}$ is the renormalized 'magnetic' length corresponding to ω_k, and where H_n is the nth Hermite polynomial. With these approximate eigenstates the current operator matrix elements read

$$\langle nk|j_x|n'k\rangle = ev_k\delta_{nn'} + e\omega_c\ell_k\left(\sqrt{(n + 1/2)}\delta_{n,n'-1} + \sqrt{n/2}\delta_{n,n'+1}\right) \tag{12}$$

$$\langle nk|j_y|n'k\rangle = -ie\omega_k\ell_k\left(\sqrt{(n + 1/2)}\delta_{n,n'-1} - \sqrt{n/2}\delta_{n,n'+1}\right) \tag{13}$$

which follows from equations (4) and (5).

3. Weak impurity scattering in the SCBA

In the conventional method for calculating the effects of impurity scattering one averages over the impurities [15, 16]]. We make the standard assumption (see e.g. [17]) that the impurity potential, V_i, is Gaussian-distributed with a zero mean and with the following impurity correlator:

$$\langle V_i(\boldsymbol{r})V_i(\boldsymbol{r}')\rangle_{\text{imp}} = \mathcal{V}(\boldsymbol{r}-\boldsymbol{r}') \tag{14}$$

where $\mathcal{V}(\boldsymbol{r})$ is an arbitrary function which we shall later take to be proportional to a delta-function. The impurity averaged Green's function G satisfies the Dyson equation

$$(E - H(\boldsymbol{r}))G(\boldsymbol{r},\boldsymbol{r}';E) - \int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{r}''\, \Sigma(\boldsymbol{r},\boldsymbol{r}'';E)G(\boldsymbol{r}'',\boldsymbol{r}';E) = \delta(\boldsymbol{r}-\boldsymbol{r}') \tag{15}$$

where the self-energy in the single-site approximation (see figure 1 and [17]) reads

$$\Sigma(x,x';E) = G(x,x';E)\mathcal{V}(x,x'). \tag{16}$$

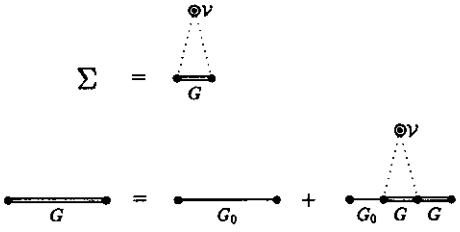

Figure 1. The impurity averaged self-energy, Σ, in the single-site approximation and the Dyson equation for the impurity averaged one-particle Green's function, G, in the SCBA.

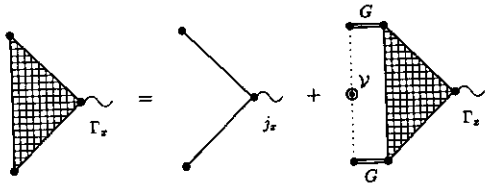

Figure 2. The integral equation for the impurity averaged vertex correction Γ_x consistent with the single-site SCBA.

In the rest of the paper, we will be working in the clean limit, i.e., the limit of a weak scattering potential. Using the $|Nk)$-basis which diagonalizes $\mathcal{H}$, and employing the SCBA, Dyson's equation takes the form

$$G(NN'k;E) - G_0(Nk;E)\sum_{N''}\Sigma(NN''k;E)G(N''N'k;E) = G_0(Nk;E)\delta_{NN'}. \tag{17}$$

It is obvious that off-diagonal elements of G are of order Σ, and to leading order in the impurity potential we therefore get $G(NN') \sim \delta_{NN'}$. Consequently,

$$G(Nk;E) = 1/[G_0^{-1}(Nk;E) - \Sigma(Nk;E)] \tag{18}$$

where, within the same set of approximations, the self-consistency equation for the self-energy reads

$$\Sigma(Nk;E)=\sum_{Mk'}\mathcal{V}(NNk,MMk')G(Mk';E). \tag{19}$$

Here $\mathcal{V}(NN'k,MM'k')$ is given by

$$\mathcal{V}(NN'k,MM'k')=\int \mathrm{d}^2r\mathrm{d}^2r'\,\mathcal{V}(r-r')\langle Nk|r\rangle\langle r|Mk'\rangle\langle M'k'|r'\rangle\langle r'|N'k\rangle. \tag{20}$$

For the imaginary part of the (retarded) self-energy we have

$$\operatorname{Im}\Sigma(Nk;E+\mathrm{i}\delta)=-\tfrac{1}{2}\sum_{Mk'}\mathcal{V}(NNk,MMk')A(Mk';E) \tag{21}$$

where A is the usual spectral function. In the clean limit A will be peaked at $E-E_{Mk'}$, and we can perform the k'-integral and get

$$\operatorname{Im}\Sigma(Nk)\simeq-\frac{L}{2\hbar}\sum_{M\pm}\frac{\mathcal{V}(NNk,MMk_M^{\pm})}{|v_{Mk_M^{\pm}}|}. \tag{22}$$

Here $k_M^{\pm}$ are the k-values which solve $E-E_{Mk}=0$. This result needs some explanation. The self-energy obtained above is, of course, nothing but what one gets form the usual Born approximation. The Born approximation cannot usually be applied in the case of a high magnetic field because of the large degeneracy of the Landau levels. However, the confinement potential lifts this degeneracy, and there are problems with divergences only at the singular points $v_k=0$. At these points the density of states diverges, but the singularity can be integrated.

However, the divergent density of states at the centre of the channel will result (at zero temperature) in discontinuities in the conductivity as the Fermi level hits the bottom of a Landau level. Therefore we should use a correct self-consistently determined Green's function at the flat part of the band in order to get rid of these unphysical features. The diagonal approximation for G, equation (18), will, however, still be valid in the clean limit, and one just has to solve equation (19) with the full expression for the Green's function in equation (18). In section 4.2 where we consider a strong confinement potential, we neglect this complication and simply use the Born approximation result equation (22).

4. Magnetoconductivity

We use the Kubo formula to calculate the magnetoconductivity, and we must therefore average over products of Green's functions. This can be done consistently only by taking vertex corrections into account. In physical terms the vertex corrections describe the difference between large- and small-angle scattering. Consequently the vertex corrections are of crucial importance for the discussion of the rôle of the spatially separated forward and backward current-carrying edge states. Furthermore, the vertex corrections are tightly connected to current conservation. Some mathematical details of this are given in appendix A.

In the case of delta-function impurities ($\mathcal{V}(\boldsymbol{r}) = 2\pi\ell^2 V_0^2 \delta(\boldsymbol{r})$), studied by Ando and others, the vertex corrections vanish in zero magnetic field and in non-zero magnetic field for an unconfined system, because all scattering angles are equally likely. However, this is *not* the case here, the reason being the spatial separation between states with different and, in particular, opposite velocities. In order to diminish the current an electron in an edge state must be scattered from one edge to the other, and therefore large and small angle scattering have indeed very different probabilities, regardless of the form of the impurity potential. If we were to calculate the conductivity neglecting the vertex corrections, we would get the incorrect answer that back-scattering always would be possible with some rate given by the average impurity potential. The correct answer is, as we shall show, given by an overlap between opposite edge states and thus strongly dependent on the width of the channel.

The conductivity formula is calculated in appendix B. Here we write the longitudinal conductivity in the clean limit using the basis where G is diagonal:

$$\sigma_L = \frac{\hbar}{2\pi L}\,\mathrm{Re}\int_{-\infty}^{\infty} \mathrm{d}\epsilon \left(-\frac{\mathrm{d}n_{\mathrm{F}}(\epsilon)}{\mathrm{d}\epsilon}\right) \sum_{NN'k} \langle N'k|j_x|Nk\rangle \times\{G(Nk;+)\Gamma_x(NN'k;+-) - G(Nk;-)\Gamma_x(NN'k;--)\}G(N'k;-) \tag{23}$$

where $G(\pm) = G(\epsilon \pm \mathrm{i}\delta)$ and similarly $\Gamma_x(\pm,-) = \Gamma_x(\epsilon \pm \mathrm{i}\delta, \epsilon - \mathrm{i}\delta)$. The vertex function, Γ_x, obeys the integral equation given in appendix A and illustrated in figure (2). Here it is written in the $|Nk\rangle$ basis:

$$\Gamma_x(NN'k;\pm,-) = \langle Nk|j_x|N'k\rangle + K_{NN'k}(\Gamma_x) \tag{24}$$

where the integral operator K is defined by

$$K_{NN'k}(\Gamma_x) = \sum_{MM'k'} \mathcal{V}(NN'k, MM'k')G(Mk';\pm) \times \Gamma_x(MM'k';\pm,-)G(M'k';-). \tag{25}$$

4.1. Conductivity and vertex corrections for a weak confinement potential

To emphasize the effect of the existence of the edge states, we initially analyse the rather unphysical limit of an extremely weak confinement potential. Somewhat unexpectedly such an approach gives an outcome very different from that of the homogeneous case [18, 11, 12]. The conductivity is simply found to be infinitely large. The physical reason for this result is that the edge states become totally decoupled and thus no scattering can take place.

Take for simplicity a parabolic confinement potential, for which γ_k, ω_k, and ℓ_k become constants and therefore are written as γ, ω, and ℓ in the following. The confinement potential can be made arbitrarily weak by letting γ approach zero independently from any of the other parameters of the problem. Consequently it is possible to have such a small value of γ that $G(nk)$ for small values of n is approximately constant as a function of $k\ell$ on the scale set by the dimensionless effective cyclotron radius $\sqrt{2n+1}$. Hence, whenever the product $G(nk)\phi_n(y - y_k)$

appears in the calculations, it can be replaced by $G(nk_y)\phi_n(y - y_k)$, where k_y is the value of k which solves $y - y_k = 0$. Since $y_k = k\ell_c^2/(1+\gamma)$ we have $k_y = y(1+\gamma)/\ell_c^2$. The discrepancy between the homogeneous sample (where vertex corrections vanish) and the case with a confinement potential (where vertex corrections are present) is most clear for delta-function impurities. We therefore consider that case here. With $\mathcal{V}(\boldsymbol{r}) = 2\pi\ell^2 V_0^2\delta(\boldsymbol{r})$, we see that $G(nn'k) \sim \delta_{nn'}$ because $\Sigma(nk)$, like $G(nk)$, is assumed to be constant within an effective cyclotron radius. The self-consistency equation (19) for the self-energy then reduces to

$$\Sigma(y;E) \simeq V_0^2 L(1+\gamma)\sum_n G(nk_y;E). \tag{26}$$

There exists the following simple relation between the $\{|nk\rangle\}$- and the $\boldsymbol{r}$-representation of the self-energy: $\Sigma(y) = \Sigma(nk_y)$, which we shall use below.

Next we turn to the vertex function. As shown in equations (4) and (12) the free vertex separates into two parts, a bulk-like and an edge-like part: $\langle nk|j_x|n'k\rangle = j_x^{\mathrm{B}}(nn'k) + j_x^{\mathrm{E}}(nn'k)$. As the start of an iterative procedure to solve for $\Gamma_x(+-)$ in equation (24) we use the definition of the integral operator $K_{nn'k}$ from equation (25) to calculate its value when it acts on j_x^{B} and j_x^{E}. We obtain $K_{nn'k}(j_x^{\mathrm{B}}) \approx 0$, a result which is exact for a system without confinement, and $K_{nn'k}(j_x^{\mathrm{E}}) = Q_k(j_x^{\mathrm{E}}(nn'k) + \gamma j_x^{\mathrm{B}}(nn'k))$ where we have defined

$$Q_k = V_0^2 L(1+\gamma)\sum_n G(nk;+)G(nk;-). \tag{27}$$

Formally the iteration procedure can be carried out to infinite order yielding the final result for the vertex function:

$$\Gamma_x(nmk) = \left(1 + \gamma\frac{Q_k}{1-Q_k}\right) j_x^{\mathrm{B}}(nmk) + \frac{1}{1-Q_k}\, j_x^{\mathrm{E}}(nmk). \tag{28}$$

This is an interesting result. Without the confinement potential, $\gamma = 0$, we have the free vertex for the homogeneous case in agreement with the fact that vertex corrections vanish in this limit, and we recover Ando and Uemura's result [11, 12]. If we put $Q_k = 0$ in the above expression we get the free vertex $\Gamma_x = j_x$. But in the clean limit Q_k is in fact equal to unity: using equation (26) and (27) we obtain,

$$\begin{aligned} Q_k &= V_0^2 L(1+\gamma)\sum_n \frac{\operatorname{Im} G(nk;+)}{\operatorname{Im}\Sigma(nk;+)} \\ &\simeq V_0^2 L(1+\gamma)\frac{1}{\operatorname{Im}\Sigma(y_k;+)}\sum_n \operatorname{Im} G(nk;+) = 1 \end{aligned} \tag{29}$$

with the drastic consequence that the vertex function diverges. This, however, is in agreement with the physical intuition that in the clean limit, when the confinement potential is very weak, the edge states become totally decoupled and thus no scattering can take place between them. In a broad, sufficiently irregular and dirty sample propagation along the edges is disrupted by the frequent scattering events, and the conductance is dominated by the diffusion processes in the bulk, leading to the

relevance of Ando and Uemura's bulk analysis. In the next section we study the case of a strongly confined clean system, where our analysis is more relevant.

The calculation in this section is based on the SCBA. The SCBA is known (in the homogeneous case) to be correct to leading order in a $1/n$-expansion, n being the number of Landau levels [19]. However, we do not believe that the inclusion of complicated quantum interference effects can alter the conclusion obtained above, because the possibility for backscattering is governed by exponentially small wave function overlaps.

4.2. Conductivity and vertex corrections for a strong confinement potential

Solving for the vertex function Γ_x in equation (24) we meet in the integral equation products of Green's functions like $G(Mk';+)G(M'k';+)$ and $G(Mk';+)G(M'k';-)$. We note that since the self-energy goes to zero in the clean limit, the particular combination $G(Mk';+)G(Mk';-)$ diverges as $1/\operatorname{Im}\Sigma(Mk')$, whereas all other combinations remain finite. With these remarks we can write down the equation for the leading correction $\Gamma(+-)$:

$$\Gamma_x(NN'k;+-) = \langle Nk|j_x|N'k\rangle + \sum_{Mk'} \mathcal{V}(NN'k, MMk') \frac{A(Mk;\epsilon)}{-2\operatorname{Im}\Sigma(Mk';+)} \Gamma_x(MMk';+-). \tag{30}$$

In the weak scattering limit the spectral function approaches $2\pi\delta(E_{Mk'} - \epsilon)$, we can therefore integrate over k' in (30) and obtain:

$$\Gamma_x(NN'k;+-) = \langle Nk|j_x|N'k\rangle - \frac{L}{2\hbar} \sum_{M\pm} \frac{\mathcal{V}(NN'k, MMk_M^{\pm})\Gamma_x(MMk_M^{\pm};+-)}{|v_{k_M^{\pm}}|\operatorname{Im}\Sigma(Mk_M^{\pm},+)} \tag{31}$$

where $k_M^{\pm}$ are the k-values which solve $\epsilon - E_{Mk} = 0$. There are two solutions, and denoted by the superscript $\pm$, for each M, corresponding to opposite drift velocities.

For $\Gamma(++)$ we get simply the free vertex because the correction is of order Σ, as can be seen from the special Ward identities derived in appendix A. The same is also true for Γ_y which means that the conductivity in the transverse direction reduces to an expression very similar to that obtained by Ando and others. This is clear because the edge states do not contribute to the current in the transverse direction. Also the result for the Hall response is similar to that occurrring in the homogeneous case.

With the same set of remarks we can easily find the most divergent contribution to σ_L which we will call σ_L^E because it originates from the current-carrying edge states. We obtain

$$\sigma_L^E = \frac{1}{2\pi} \sum_N \langle Nk_N|j_x|Nk_N\rangle \int_{-\infty}^{\infty} d\epsilon \left(-\frac{\partial n_F(\epsilon)}{\partial\epsilon}\right) \frac{(-\Gamma_x(NNk_N,+-))}{|v_{Nk_N}|\operatorname{Im}\Sigma(Nk_N,+)}. \tag{32}$$

We now proceed to solve for $\Gamma_x(+-)$. First we find the diagonal piece of Γ_x which, when inserted back into (31), also gives the off-diagonal parts. If we make the *ansatz*

$$\Gamma_x(NNk_N^i,+-) = a_N\langle Nk_N^i|j_x|Nk_N^i\rangle \tag{33}$$

where i labels the two k-solutions for each N, we obtain the solution

$$a_N = \sum_M \left\{[\mathbf{I} - \mathbf{C}]^{-1}\right\}_{NM} \tag{34}$$

where

$$C_{NM} = \frac{L}{2\hbar} \sum_{\pm} \mathcal{V}(NNk_N^i, MMk_M^{\pm}) \frac{-\mathrm{sgn}(v_{Mk_M^{\pm}})}{v_{Nk_N^i} \operatorname{Im} \Sigma(Mk_M^{\pm})}. \tag{35}$$

The route is now clear: we must calculate the elements of the matrix $\mathbf{C}$ and then insert the expression for the vertex function into the conductivity formula. This, however, cannot be done in general, and therefore we take the simplifying example of a parabolic confinement potential. However, we emphasize that a similar calculation can be done in the general case when utilizing the high-field approximation for the eigenstates made in section 2. As previously, we drop the subscript k for the constant parameters γ_k, ω_k, and ℓ_k defined in section 2. The eigenstates of $\mathcal{H}_k$ are

$$\langle \boldsymbol{r} | nk \rangle = (\mathrm{e}^{\mathrm{i}kx}/\sqrt{L})\phi_n(y - y_k) \tag{36}$$

where ϕ_n is the usual harmonic oscillator solution with the renormalized magnetic length ℓ and the renormalized oscillator frequency ω. Furthermore, we have the relations $y_k = k\ell_c^2/(1+\gamma)$ and $v_k = \gamma y_k \omega_c$.

For simplicity we study only short-range impurity scattering described by the impurity correlator $\mathcal{V}(\boldsymbol{r})$:

$$\mathcal{V}(\boldsymbol{r}) = 2\pi\ell^2 V_0^2 \delta(\boldsymbol{r}). \tag{37}$$

We define the dimensionless overlap function F_{mj}^{ni} by

$$F_{mj}^{ni} = \sqrt{2\pi}\ell \int_{-\infty}^{\infty} \mathrm{d}y \left|\phi_n(y - y_{k_n^i})\right|^2 \left|\phi_m(y - y_{k_m^j})\right|^2 \tag{38}$$

where $i, j = \pm$. The expressions for $\operatorname{Im} \Sigma(nk)$ and the matrix $\mathbf{C}$ then have the form

$$\operatorname{Im} \Sigma(nk) = -\sqrt{\frac{\pi}{2}} \frac{\ell}{\hbar} V_0^2 \sum_{mj} F_{n+}^{mj} \Big/ |v_{mk_m^j}| \tag{39}$$

$$C_{nm} = \frac{1}{v_{k_n^+}} \sum_i F_{n+}^{mi} \mathrm{sgn}(v_{k_m^i}) \Big/ \left(\sum_{m'j} F_{mi}^{m'j} \Big/ |v_{k_{m'}^j}| \right). \tag{40}$$

In this case the conductivity becomes

$$\sigma_{\mathrm{L}}^{\mathrm{E}} = \sigma_0 \sum_n \int_{-\infty}^{\infty} \mathrm{d}\epsilon \left(-\frac{\mathrm{d}n_{\mathrm{F}}(\epsilon)}{\mathrm{d}\epsilon} \right) |v_{k_n}| a_n \Big/ \left(\omega^2 \ell^2 \sum_{mj} F_{n+}^{mj} / |v_{k_m^j}| \right) \tag{41}$$

where

$$\sigma_0 = \frac{e^2}{h} \left(\frac{\hbar\omega}{V_0} \right)^2 \sqrt{\frac{2}{\pi}} \ell. \tag{42}$$

In order to find analytical expressions for σ_L^E, we restrict ourselves here to the case where the Fermi level crosses one or two Landau levels. This is also a generic case which enables us to study what happens when the Fermi level sweeps through a new level. When only one Landau level participates in the conduction of current, the conductivity formula at zero temperature becomes particularly simple:

$$\sigma_L^E = \sigma_0 (v_{k_0^+}/\omega\ell)^2 / 2F_{0-}^{0+} = \sigma_0 (v_{k_0^+}/\omega\ell)^2 \tfrac{1}{2} \exp(2y_{k_0^+}^2/\ell^2). \tag{43}$$

The conductivity thus increases exponentially with the width of the sample. This is very reasonable because backscattering is proportional with the exponentially decreasing overlap of the wave function of opposite edge states.

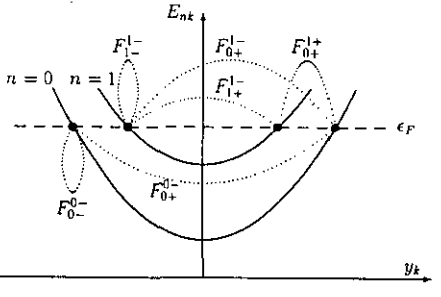

Figure 3. All six basically different inter- and intra-Landau-level scattering processes at the Fermi energy involving the two lowest Landau levels of the quantum wire.

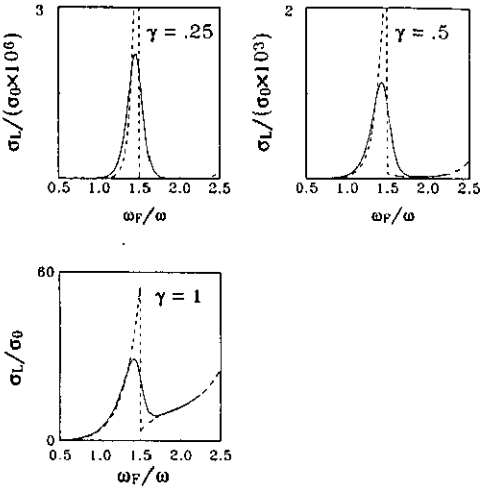

Figure 4. The longitudinal conductivity, σ_L^E, shown as a function of the Fermi frequency, $\omega_F = \epsilon_F/\hbar$, for $\gamma = 0.25, 0.50$ and 1.00. The dashed lines are the zero-temperature results, while full lines correspond to $k_B T = 0.05\hbar\omega$.

When two Landau levels participate in the conduction of current, the situation is more complicated. There are now six different possible scattering processes as shown in figure (3). Utilizing the relations $F_{mj}^{ni} = F_{ni}^{mj}$ and $F_{mj}^{ni} = F_{m(-j)}^{n(-i)}$, equation (41) yields the following expression for the conductivity at zero temperature:

$$\sigma_L^E = \sigma_0 \left(\frac{v_{k_0^+}}{\omega\ell}\right)^2 \frac{2F_{0+}^{1+} + F_{0+}^{0-}/x + xF_{1+}^{1-}}{2x(F_{0+}^{1-}F_{0+}^{1+} + F_{0+}^{0-}F_{1+}^{1-}) + (F_{0+}^{0-} + x^2 F_{1+}^{1-})(F_{0+}^{1-} + F_{0+}^{1+})} \tag{44}$$

where $x = v_{k_0^+}/v_{k_1^+}$. The numerical results presented in figure (4) show the conductivity for three values of γ and for two different temperatures as a function of the Fermi energy.

As can be seen in the figure the conductivity rises quickly as the Fermi level increases above $\frac{1}{2}\hbar\omega$. This is of course due to the increasing separation between the

two edge states, in accordance with equation (43). When ϵ_F hits the next Landau level, the conductivity starts to decrease because a channel for backscattering is opened, i.e., the $n = 0$ edge state can scatter to the opposite edge through the $n = 1$ state. However, when opposite states of the $n = 1$ band become decoupled, the conductivity starts to increase again. This qualitative behaviour is repeated each time a new Landau level crosses the Fermi level. The height of the conductance peak at $\omega_F = \frac{3}{2}\omega$ increases strongly with increasing channel width.

5. Summary and conclusions

We have developed a formalism for calculating the magnetoconductivity of disordered quantum wires in high magnetic fields. Based on the generalized Ward identity, which connects the impurity-averaged Green's function and vertex function, the theory consistently incorporates the important difference between backward and forward scattering: only backscattering diminishes the current. Even for short-range scatters the presence of the confinement potential leads to a non-zero vertex correction. This is in contrast to the bulk electron gas both in zero and finite magnetic field. The reason for this difference is the development of spatially separated current-carrying edge channels. In close analogy with the well known difference between scattering time and transport time in the case of zero magnetic field and finite-ranged impurity scattering, the essential physics in our problem is contained in the vertex corrections.

An explicit calculation of the magnetoconductivity is carried out in the weak scattering limit. There the Born approximation is applicable, except near the bottom of the Landau levels where divergences appear. However, the resulting unphysical discontinuities in the conductivity are smeared out by the inclusion of a finite temperature. We have shown examples of the specific case where only two Landau levels cross the Fermi energy. Generalization of the numerical calculation to include more Landau levels is straightforward.

The analogy between our expression for the conductivity in high fields and the Boltzmann equation result for the conductivity in the zero magnetic field suggests that a similar approach could be applied to the strong magnetic field case. In the Boltzmann equation analogy the interpretation of a_N in equation (34) is related to the ratio between the scattering time and the transport time. Indeed, the results derived in the present paper can also be obtained from a modified Boltzmann equation approach [13, 14], which will be dealt with in a later publication.

As an experimental test of the theory we suggest studies of weakly disordered quantum wires in strong magnetic fields. This could, e.g., be realized by applying a long split gate confining a relatively high mobility two-dimensional electron gas. The high-mobility is necessary for having well defined Landau quantization. On the other hand the length of the wire should exceed the mean free path of the electrons, in order to have well defined local conductivity, which is the quantity that we calculate.

Appendix A. Ward identity

In order to make a conserving approximation for the current–current correlation function, we must ensure that the so-called generalized Ward identity is fulfilled. Here we first derive the identity and then show its consequences for the vertex function.

We follow the procedure of Nambu [20, 21]. Define the function

$$\Lambda_\mu(x,y,z) = \langle \mathcal{T}\{j_\mu(z)\Psi(x)\Psi^\dagger(y)\}\rangle \tag{A1}$$

where we use the four-vector (or in our case three-vector since we are considering two spatial dimensions) notation. Thus $x = (i\tau, r)$ and j is the four-vector current density operator, where j_0 is given by $e\rho$, and ρ is the density operator. The vertex function Γ is defined through

$$\Lambda_\mu(x,y,z) = \int d^4x' \int d^4y'\, G(x,x')\Gamma_\mu(x',y',z)G(y',y) \tag{A2}$$

where

$$G(x,x') = \langle \mathcal{T}\{\Psi(x)\Psi^\dagger(x')\}\rangle . \tag{A3}$$

By applying the four-vector gradient operator on the function Λ, we get, after the use of the continuity equation,

$$\sum_{\mu=1}^{3}\left[\partial_{z_\mu} j_\mu(z) - i\partial_{z_0} j_0(z)\right] = 0 \tag{A4}$$

and the following relation between G and Γ

$$G(x,y)[\delta(y,z) - \delta(x,z)] = \int d^4x' \int d^4y'\, G(x,x') \times \left(\sum_{\mu=1}^{3}\partial_{z_\mu}\Gamma_\mu(x',y',z) - i\partial_{z_0}\Gamma_0(x',y',z)\right)G(y',y) \tag{A5}$$

which can also be written as

$$i\delta(x,z)G^{-1}(x,y) - iG^{-1}(x,y)\delta(y,z) = \left(\sum_{\mu=1}^{3}\partial_{z_\mu}\Gamma_\mu(x,y,z) - i\partial_{z_0}\Gamma_0(x,y,z)\right). \tag{A6}$$

This relation is the generalized Ward identity. If this relation is satisfied in the approximation, we are guaranteed gauge-invariance and current-conservation which, of course, are necessary.

We now show what the appropriate vertex function is, for the self-consistent Born approximation. In the lowest order SCBA the self-energy is given by

$$\Sigma(x,x') = G(x,x')\mathcal{V}(x,x') \tag{A7}$$

where $\mathcal{V}(x-x') = \langle V(r)V(r')\rangle$ and, in the case of impurity scattering, is independent of time. We can, however, quite generally show that if the vertex function satisfies the integral equation:

$$\Gamma_\mu(x,y,z) = \Gamma^0_\mu(x,y,z) + \mathcal{V}(x,y)\int d^4x' \int d^4y'\, G(x,x')\Gamma_\mu(x',y',z)G(y',y) \tag{A8}$$

it is in accordance with the generalized Ward identity. In (A8) Γ^0 is the free vertex.

We now shift to the basis $\{|\alpha\rangle\}$ which diagonalizes G, and we have

$$\Lambda_\mu(x,y,z) = \sum_{\alpha\beta}\langle x|\alpha\rangle\langle\beta|y\rangle\Lambda_\mu(\alpha,\beta,z;x_0,y_0). \tag{A9}$$

We also define the Fourier transform with respect to time as

$$\Lambda(x_0,y_0,z_0) = \frac{1}{\beta^2}\sum_{\omega\omega'} e^{i\omega'(x_0-y_0)/\hbar+i\omega(x_0-z_0)/\hbar}\Lambda(i\omega'+i\omega,i\omega'). \tag{A10}$$

Here $i\omega, i\omega'$ are Matsubara frequencies. The definition of the vertex function reads, in this basis,

$$\Lambda(z,\alpha,\beta;i\omega'+i\omega,i\omega') = G(\alpha;i\omega'+i\omega)\Gamma_\mu(z,\alpha,\beta;i\omega+i\omega',i\omega')G(\beta,i\omega') \tag{A11}$$

with the convention $z = (z_1, z_2)$ (when working in two dimensions). Similarly, we have for the generalized Ward identity

$$i\langle\alpha|z\rangle\langle z|\beta\rangle\left[G^{-1}(\beta;i\omega') - G^{-1}(\alpha;i\omega+i\omega')\right] = \sum_{\mu=1}^{2}\partial_{z_\mu}\Gamma_\mu(z,\alpha,\beta;i\omega+i\omega',i\omega') - \frac{\omega}{\hbar}\Gamma_0. \tag{A12}$$

In order to extract an equation relating the vertex function when integrated over z to the Green's function, we take the Fourier transform of (A12) with respect to z. This is useful because the conductivity is given by the average over z. We have

$$\langle\alpha|e^{iq\cdot z}|\beta\rangle\left[G^{-1}(\beta;i\omega') - G^{-1}(\alpha;i\omega+i\omega')\right] = \sum_{\mu=1}^{2} q_\mu\Gamma_\mu(q,\alpha,\beta,i\omega+i\omega',i\omega') + \frac{i\omega}{\hbar}\Gamma_0. \tag{A13}$$

We are interested in the limit $q \to 0$. However, we can only express Γ_μ in terms of the Green's functions if we simultaneously take the limit $\omega \to 0$. Consequently the Ward identity is useful for finding $\Gamma(\epsilon+i\delta,\epsilon+i\delta)$ only and cannot be used for evaluation of e.g. $\Gamma(\epsilon+i\delta,\epsilon-i\delta)$, where the arguments are different [16]. In these limits we get

$$\partial_{q_\mu}\langle\alpha|e^{iq\cdot z}|\beta\rangle\Big|_{q=0}\left[G^{-1}(\beta;i\omega') - G^{-1}(\alpha;i\omega')\right] = \Gamma_\mu(q=0,\alpha,\beta,i\omega',i\omega'). \tag{A14}$$

In our specific case where we use the Landau gauge, the system is translationally invariant in the x-direction. The eigenbasis can therefore be labelled as $|\alpha\rangle = |Nk\rangle$, where

$$\langle z|\alpha\rangle = e^{ikx}\langle y|Nk\rangle \tag{A15}$$

using $z = (x, y)$. It is then a matter of algebra to extract the vertex function from the above equation. We obtain

$$\Gamma_x(Nk, N'k'; i\omega, i\omega) = \Gamma_x^B + \Gamma_x^E \tag{A16}$$

$$\Gamma_x^B(Nk, N'k'; i\omega, i\omega) = -\frac{e}{\hbar}[G^{-1}(Nk; i\omega) - G^{-1}(N'k; i\omega)]\langle Nk|\partial/\partial k|N'k\rangle\, \delta_{kk'} \tag{A17}$$

$$\Gamma_x^E(Nk, N'k'; i\omega, i\omega) = -\frac{e}{\hbar}\frac{\partial}{\partial k}G^{-1}(Nk, i\omega)\, \delta_{kk'}\delta_{NN'} \tag{A18}$$

$$\Gamma_y(Nk, N'k'; i\omega, i\omega) = -i\frac{e}{\hbar}\left[G^{-1}(Nk; i\omega) - G^{-1}(N'k; i\omega)\right]\langle Nk|y|N'k\rangle\, \delta_{kk'}. \tag{A19}$$

Here Γ means the $q = 0$ part. The two terms in Γ_x may be interpreted as coming from bulk and edge states, respectively. The first term has to do with scattering from one Landau level to another, whereas the latter describes transport within the same Landau level with a weight proportional to the group velocity.

Equations (A16)–(A19) are in fact very general. They do not depend on any particular type of interaction or impurity model. As long as we can find the self-energy, we have automatically the vertex function. It is, however, as mentioned, restricted to the $i\omega = 0$ part.

Appendix B. Conductivity formula

The conductivity of the wire will be calculated by means of the Kubo formula. The current in the x-direction is given by the integrated current density,

$$I_x = \frac{1}{L}\int_{-L/2}^{L/2} dx \int_{-\infty}^{\infty} dy\, \langle j_x(\boldsymbol{r})\rangle \tag{B1}$$

where $j_x(\boldsymbol{r})$ is the local current operator. Here we have averaged the current over the length L of the wire. The averaging is unimportant since the current is independent of x. In terms of the conductivity tensor the current density is

$$\langle j_x(\boldsymbol{r})\rangle = \int d^2\boldsymbol{r}'\, \sigma(\boldsymbol{r}, \boldsymbol{r}')E_x(\boldsymbol{r}'). \tag{B2}$$

Taking $E_x(\boldsymbol{r})$ to be slowly varying and given by V/L, we get the final expression for the longitudinal conductivity in the usual way from the Kubo formula:

$$g_L = -\lim_{\omega\to 0}\left[\frac{\hbar}{\omega}\operatorname{Im}\Pi_L(\omega + i\delta)\right]. \tag{B3}$$

Here Π_L is the impurity averaged current–current correlation function given by

$$\Pi_L(\tau) = \langle T(I_x(\tau)I_x)\rangle. \tag{B4}$$

The relation between the longitudinal conductance and the longitudinal conductivity is, for this one-dimensional system, simply given by

$$\sigma_{\mathrm{L}} = L g_{\mathrm{L}}. \tag{B5}$$

The conductivity defined above is the local, spatially averaged conductivity that one would measure if a macroscopic measurement with ideal voltage probes were performed. The formula is thus not applicable to mesoscopic wires as it does not take into account the contacts and the finite size of the scattering region. In the case of a mesoscopic sample the appropriate formula for the conductance is instead given by [22–24]

$$g_{\mathrm{L}} = -\int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}y' \sigma(0, y; L_{\mathrm{s}}, y') \tag{B6}$$

which leads to the Landauer formula relating the conductance to the transmission probability. Here ideal reservoirs are thought to be connected to the wire at $x = 0$ and $x = L_{\mathrm{s}}$. An interesting question is how the Landauer formula is modified if an impurity averaging is performed. The methods used in the present paper could also be used to derive such a result, but the lack of translational invariance would greatly complicate the matters.

The current–current correlation function can now be written as

$$\Pi_{\mathrm{L}}(\mathrm{i}\omega) = \frac{1}{L^2\beta} \sum_{\omega'} \sum_{\alpha\beta} \langle\beta|j_x|\alpha\rangle P_x(\alpha, \beta; \mathrm{i}\omega + \mathrm{i}\omega', \mathrm{i}\omega') \tag{B7}$$

where P_x can be expressed in term of the vertex function defined in appendix A,

$$P_x(\alpha, \beta; \mathrm{i}\omega + \mathrm{i}\omega', \mathrm{i}\omega') = G(\alpha, \mathrm{i}\omega + \mathrm{i}\omega')\Gamma_x(\alpha, \beta; \mathrm{i}\omega + \mathrm{i}\omega', \mathrm{i}\omega')G(\beta, \mathrm{i}\omega'). \tag{B8}$$

The Matsubara sum in (B7) is evaluated in the standard fashion by a contour integral which has two branch cuts [16]. The result of the contour integration reads after the analytic continuation $\mathrm{i}\omega \to \omega + \mathrm{i}\delta$:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\epsilon}{2\pi\mathrm{i}} n_{\mathrm{F}}(\epsilon)[P(\epsilon + \omega + \mathrm{i}\delta, \epsilon + \mathrm{i}\delta) - P(\epsilon + \omega + \mathrm{i}\delta, \epsilon - \mathrm{i}\delta) + P(\epsilon + \mathrm{i}\delta, \epsilon - \omega - \mathrm{i}\delta) - P(\epsilon - \mathrm{i}\delta, \epsilon - \omega - \mathrm{i}\delta)]. \tag{B9}$$

This can be rearranged to give, after dividing with ω and taking the limit $\omega \to 0$,

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\epsilon}{2\pi\mathrm{i}} \left(-\frac{\mathrm{d}n_{\mathrm{F}}(\epsilon)}{\mathrm{d}\epsilon}\right)[P(\epsilon + \mathrm{i}\delta, \epsilon - \mathrm{i}\delta) - P(\epsilon - \mathrm{i}\delta, \epsilon - \mathrm{i}\delta)] + \int_{-\infty}^{\infty} \frac{\mathrm{d}\epsilon}{2\pi\mathrm{i}} n_{\mathrm{F}}(\epsilon) \times \left[\frac{\partial}{\partial\omega}\{P(\epsilon + \omega + \mathrm{i}\delta, \epsilon + \mathrm{i}\delta) - P(\epsilon + \omega - \mathrm{i}\delta, \epsilon - \mathrm{i}\delta)\}\right]_{\omega=0}. \tag{B10}$$

In the basis that we will be working in, the free vertex is always purely real or purely imaginary. For the diagonal part of the conductance we thus get a contribution from the first term in (B10) only. (For the Hall response it is only the last term that contributes.) We obtain

$$\sigma_{\mu\mu} = \frac{\hbar}{2\pi L} \int_{-\infty}^{\infty} \mathrm{d}\epsilon \left(-\frac{\mathrm{d}n_{\mathrm{F}}(\epsilon)}{\mathrm{d}\epsilon}\right) \mathrm{Re} \sum_{\alpha\beta} \langle\beta|j_\mu|\alpha\rangle \times \{G(\beta, +)\Gamma_\mu(\beta\alpha, +-) - G(\beta, -)\Gamma_\mu(\beta\alpha, --)\} G(\alpha, -) \tag{B11}$$

where $G(\pm) = G(\epsilon \pm \mathrm{i}\delta)$. This is the expression which is used in the main part of the paper.

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